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Lepton flavor violating decays of Standard-Model-like Higgs in 3-3-1 model with neutral lepton

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Abstract

The one loop contribution to the lepton flavor violating decay $h^0 \to \mu\tau$ of the SM-like neutral Higgs (LFVHD) in the 3-3-1 model with neutral lepton is calculated using the unitary gauge. We have checked in detail that the total contribution is exactly finite, and the divergent cancellations happen separately in two parts of active neutrinos and exotic heavy leptons. By numerical investigation, we have indicated that the one-loop contribution of the active neutrinos is very suppressed while that of exotic leptons is rather large. The branching ratio of the LFVHD strongly depends on the Yukawa couplings between exotic leptons and $SU(3)_L$ Higgs triplets. This ratio can reach 10^{-5} providing large Yukawa couplings and constructive correlations of the $SU(3)_L$ scale (v_3) and the charged Higgs masses. The branching ratio decreases rapidly with the small Yukawa couplings and large v_3.

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1. Introduction

The observation the Higgs boson with mass around 125.09 GeV by experiments at the Large Hadron Collider (LHC) [1–5] again confirms the very success of the Standard Model (SM) at low energies of below few hundred GeV. But the SM must be extended to solve many well-known problems, at least the question of neutrino masses and neutrino oscillations which have been experimentally confirmed [6]. Neutrino oscillation is a clear evidence of lepton flavor violation in the neutral lepton sector which may give loop contributions to the rare lepton flavor violating (LFV) decays of charged leptons, Z and SM-like Higgs bosons. Therefore, these are the promoting subjects of new physics which have been hunted by recent experiments [7–9]. Especially, the latest experimental results of LFVHD have been reported recently by CMS and ATLAS. Defining $\mathrm{Br}(h^0 \to \mu\tau) \equiv \mathrm{Br}(h^0 \to \mu^+\tau^-) + \mathrm{Br}(h^0 \to \mu^-\tau^+)$, the upper bound $\mathrm{Br}(h^0 \to \mu\tau) < 1.5 \times 10^{-2}$ at 95% C.L. was announced by CMS, in agreement with 1.85×10^{-2} at 95% C.L. from ATLAS. These sensitivities are not far from the recent theoretical prediction and is hoped to be improved soon, as discussed in [10].

The LFVHD of the neutral Higgses have been investigated widely in the well-known models beyond the SM [11,12,10], including the supersymmetric (SUSY) models [13–15]. The SUSY versions usually predict large branching ratio of LFVHD which can reach 10^{-4} or higher, even up to 10^{-2} in recent investigation [13], provided the two following requirements: new LFV sources from sleptons and the large $\tan\beta$-ratio of two vacuum expectation values (vev) of two neutral Higgses. At least it is true for the LFVHD $h^0 \to \mu\tau$ under the restrict of the recent upper bound of $\mathrm{Br}(\tau \to \mu\gamma) < 10^{-8}$ [16]. In the non-SUSY $SU(2)_L \times U(1)_Y$ models beyond the SM such as the seesaw or general two Higgs doublet (THDM), the LFVHD still depends on the LFV decay of τ lepton. The reason is that the LFVHD is strongly affected by Yukawa couplings of leptons while the $SU(2)_L \times U(1)_Y$ contains only small Yukawa couplings of normal charged leptons and active neutrinos. Therefore, many of non-SUSY versions predict the suppressed signal of LFVHD.

Based on the extension of the $SU(2)_L \times U(1)_Y$ gauge symmetry of the SM to the $SU(3)_L \times U(1)_X$, there is a class of models called 3-3-1 models which inherit new LFV sources. Firstly, the particle spectra include new charged gauge bosons and charged Higgses, normally carrying two units of lepton number. Secondly, the third components of the lepton (anti-) triplets may be normal charged leptons [17,18] or new leptons [19–23] with non-zero lepton numbers. These new leptons can mix among one to another to create new LFV changing currents, except the case of normal charged leptons. The most interesting models for LFVHD are the ones with new heavy leptons corresponding to new Yukawa couplings that affect strongly to the LFVHD through the loop contributions. This property is different from the models based on the gauge symmetry of the SM including the SUSY versions. In the 3-3-1 models, if the new particles and the $SU(3)_L$ scale are larger than few hundred GeVs, the one-loop contributions to the LFV decays of τ always satisfy the recent experimental bound [24]. While this region of parameter space, even at the TeV values of the $SU(3)_L$ scale, favors the large branching ratios of LFVHD. The one-loop contributions on LFV processes in SUSY versions of 3-3-1 models were given in [25,14], but the non-SUSY contributions were not mentioned.

The 3-3-1 models were first investigated from interest of the simplest expansion of the $SU(2)_L$ gauge symmetry and the simplest lepton sector [17]. They then became more attractive by a clue of answering the flavor question coming from the requirement of anomaly cancellation for $SU(3)_L \times U(1)_X$ gauge symmetry [18]. The violation of the lepton number is a natural property of these models, leading to the natural presence of the LFV processes and neutrino oscillations.

Many versions of 3-3-1 models have been constructed for explaining other unsolved questions in the SM limit: solving the strong CP problem [26] with Peccei–Quinn symmetry [27]; allowing the electric charge quantization [28]... More interesting, the neutral heavy leptons or neutral Higgses can play roles of candidates of dark matter (DM) [23]. Besides, the models with neutral leptons are still interesting for investigation of precision tests [19].

From the above reasons, this work will pay attention to the LFVHD of the 3-3-1 with left-handed heavy neutral leptons or neutrinos (3-3-1LHN) [23]. It is then easy to predict which specific 3-3-1 models can give large signals of LFVHD. As we will see, the 3-3-1 models usually contain new heavy neutral Higgses, including both CP-even and odd ones. But the recent lower bound of the $SU(3)_L$ scale is few TeV, resulting the same order of these Higgs masses. At recent collision energies of experiments, the opportunity to observe these heavy neutral Higgses seems rare. We therefore concentrate only on the SM-like Higgs.

Our work is arranged as follows. The section 2 will pay attention on the formula of branching ratio of LFVHD which can be also applied for new neutral CP-even Higgses, listing the Feynman rules and the needed form factors to calculate the amplitudes for general 3-3-1 models. In the section 3, the model constructed in [23] will be improved including adding new LFV couplings; imposing a custodial symmetry on the Higgs potential to cancel large flavor neutral changing currents in the Higgs sector and simplify the Higgs self-interactions. From this both masses and mass eigenvectors of even-CP neutral Higgses are found exactly at the tree level. The section 4 represents numerical results of LFVHD, where the most interesting region of the parameter space will be chosen based on the latest experimental results relating to lower bounds of new gauge bosons and charged Higgses. We concentrate on the roles of Yukawa couplings of exotic neutral leptons, the charged Higgses and the $SU(3)_L$ scale. We summarize our main results in the conclusion section. The appendices show notations of Passarino–Veltman functions, the detail of calculating one-loop contributions to LFVHD amplitude in the 3-3-1LHN and the divergent cancellation.

2. Formulas for decay rates of neutral Higgses

For studying the LFVHD, namely $h^0 \to \tau^{\pm}\mu^{\mp}$, we consider the general form of the corresponding LFV effective Lagrangian as follows

$$-\mathcal{L}^{LFV} = h^0 \left(\Delta_L \bar{\mu} P_L \tau + \Delta_R \bar{\mu} P_R \tau\right) + \text{h.c.}, \tag{1}$$

where $\Delta_{L,R}$ are scalar factors arisen from the loop contributions. In the unitary gauge, the one-loop diagrams contributing to $\Delta_{L,R}$ are listed in the Fig. 1. They can be applied for the models beyond the SM where the particle contents include only Higgses, fermions and gauge bosons. The amplitude decay is [10]:

$$i\mathcal{M} = -i\bar{u}_1 \left(\Delta_L P_L + \Delta_R P_R\right) v_2, \tag{2}$$

where $u_1 \equiv u_1(p_1, s_1)$ and $v_2 \equiv v_2(p_2, s_2)$ are respective Dirac spinors of the μ and τ. The partial width of the decays is

$$\begin{aligned}\Gamma(h^0 \to \mu\tau) &\equiv \Gamma(h^0 \to \mu^-\tau^+) + \Gamma(h^0 \to \mu^+\tau^-)\\ &= \frac{1}{8\pi m_{h^0}} \times \sqrt{\left[1 - \left(\frac{m_1 + m_2}{m_{h^0}}\right)^2\right]\left[1 - \left(\frac{m_1 - m_2}{m_{h^0}}\right)^2\right]}\\ &\quad \times \left[\left(m_{h^0}^2 - m_1^2 - m_2^2\right)\left(|\Delta_L|^2 + |\Delta_R|^2\right) - 4m_1 m_2 \text{Re}\left(\Delta_L \Delta_R^*\right)\right],\end{aligned} \tag{3}$$

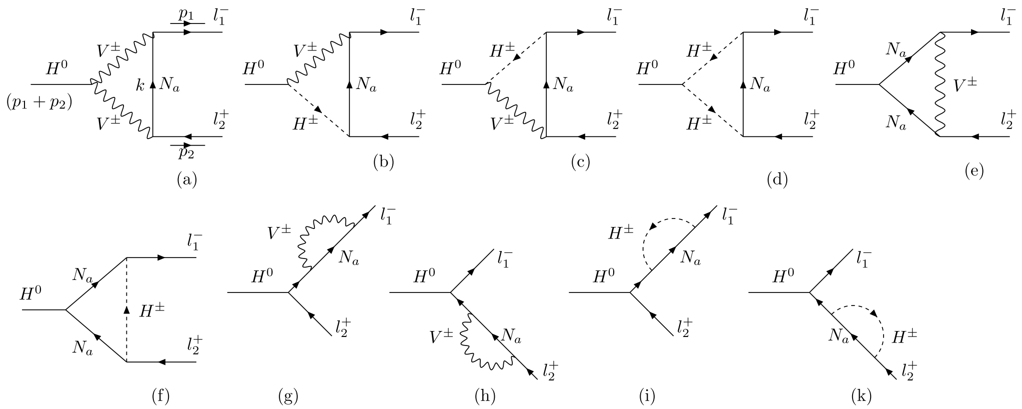

Fig. 1. Feynman diagrams contributing to the $H^0 \to \mu^\pm\tau^\mp$ decay in the unitary gauge, where H^0 is an arbitrary even-CP neutral Higgs in the 3-3-1 models, including the SM-like one.

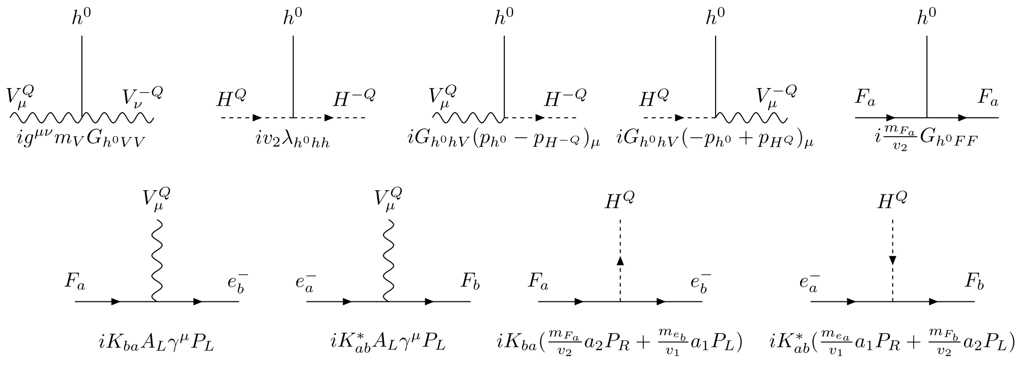

Fig. 2. Feynman rules for the $h^0 \to \mu^\pm\tau^\mp$ in the unitary gauge, where all momenta are incoming.

where m_{h^0}, m_1 and m_2 are the masses of the neutral Higgs h^0, muon and tauon, respectively. They satisfy the on-shell conditions for external particles, namely $p_i^2 = m_i^2$ $(i = 1, 2)$ and $p_0^2 \equiv (p_1 + p_2)^2 = m_{h^0}^2$.

In the unitary gauge, the relevant Feynman rules for the LFV decay of $h^0 \to l_1^\pm l_2^\mp$ are represented in the Fig. 2.

For each diagram, there is a corresponding generic function expressing its contribution to the LFVHD. These functions are defined as

$$E_L^{FVV}(m_F, m_V) = m_V m_1 \left\{ \frac{1}{2m_V^4}\left[m_F^2\left(b_1^{(1)} - b_0^{(1)} - b_0^{(2)}\right) - m_2^2 b_1^{(2)} + \left(2m_V^2 + m_{h^0}^2\right) m_F^2 \left(C_0 - C_1\right)\right] - \left(2 + \frac{m_1^2 - m_2^2}{m_V^2}\right) C_1 + \left(\frac{m_1^2 - m_{h^0}^2}{m_V^2} + \frac{m_2^2 m_{h^0}^2}{2m_V^4}\right) C_2 \right\}, \tag{4}$$

$$E_R^{FVV}(m_F, m_V) = m_V m_2 \left\{ \frac{1}{2m_V^4} \left[-m_F^2 \left(b_1^{(2)} + b_0^{(1)} + b_0^{(2)} \right) + m_1^2 b_1^{(1)} + (2m_V^2 + m_{h^0}^2) m_F^2 (C_0 + C_2) \right] + \left(2 + \frac{-m_1^2 + m_2^2}{m_V^2} \right) C_2 - \left(\frac{m_2^2 - m_{h^0}^2}{m_V^2} + \frac{m_1^2 m_{h^0}^2}{m_V^4} \right) C_1 \right\}, \quad (5)$$

$$E_L^{FVH}(a_1, a_2, v_1, v_2, m_F, m_V, m_H) = m_1 \left\{ -\frac{a_2}{v_2} \frac{m_F^2}{m_V^2} \left(b_1^{(1)} - b_0^{(1)} \right) + \frac{a_1}{v_1} m_2^2 \left[2C_1 - \left(1 + \frac{m_h^2 - m_{h^0}^2}{m_V^2} \right) C_2 \right] + \frac{a_2}{v_2} m_F^2 \left[C_0 + C_1 + \frac{m_h^2 - m_{h^0}^2}{m_V^2} (C_0 - C_1) \right] \right\}, \quad (6)$$

$$E_R^{FVH} a_1, a_2, v_1, v_2, m_F, m_V, m_H) = m_2 \left\{ \frac{a_1}{v_1} \left[\frac{m_1^2 b_1^{(1)} - m_F^2 b_0^{(1)}}{m_V^2} + \left(m_F^2 C_0 - m_1^2 C_1 + 2m_2^2 C_2 + 2(m_{h^0}^2 - m_2^2) C_1 - \frac{m_h^2 - m_{h^0}^2}{m_V^2} \left(m_F^2 C_0 - m_1^2 C_1 \right) \right) \right] + \frac{a_2}{v_2} m_F^2 \left(-2C_0 - C_2 + \frac{m_h^2 - m_{h^0}^2}{m_V^2} C_2 \right) \right\}, \quad (7)$$

$$E_L^{FHV}(a_1, a_2, v_1, v_2, m_F, m_H, m_V) = m_1 \left\{ \frac{a_1}{v_1} \left[\frac{-m_2^2 b_1^{(2)} - m_F^2 b_0^{(2)}}{m_V^2} + \left(m_F^2 C_0 - 2m_1^2 C_1 + m_2^2 C_2 - 2(m_{h^0}^2 - m_1^2) C_2 - \frac{m_h^2 - m_{h^0}^2}{m_V^2} \left(m_F^2 C_0 + m_2^2 C_2 \right) \right) \right] + \frac{a_2}{v_2} m_F^2 \left(-2C_0 + C_1 - \frac{m_h^2 - m_{h^0}^2}{m_V^2} C_1 \right) \right\}, \quad (8)$$

$$E_R^{FHV}(a_1, a_2, v_1, v_2, m_F, m_H, m_V) = m_2 \left\{ \frac{a_2}{v_2} \frac{m_F^2}{m_V^2} \left(b_1^{(2)} + b_0^{(2)} \right) + \frac{a_1}{v_1} m_1^2 \left[-2C_2 + \left(1 + \frac{m_h^2 - m_{h^0}^2}{m_V^2} \right) C_1 \right] + \frac{a_2}{v_2} m_F^2 \left[C_0 - C_2 + \frac{m_h^2 - m_{h^0}^2}{m_V^2} (C_0 + C_2) \right] \right\}. \quad (9)$$

$$E_L^{FHH}(a_1, a_2, v_1, v_2, m_F, m_H) = m_1 v_2 \left[\frac{a_1 a_2}{v_1 v_2} m_F^2 C_0 - \frac{a_1^2}{v_1^2} m_2^2 C_2 + \frac{a_2^2}{v_2^2} m_F^2 C_1 \right], \tag{10}$$

$$E_R^{FHH}(a_1, a_2, v_1, v_2) = m_2 v_2 \left[\frac{a_1 a_2}{v_1 v_2} m_F^2 C_0 + \frac{a_1^2}{v_1^2} m_1^2 C_1 - \frac{a_2^2}{v_2^2} m_F^2 C_2 \right], \tag{11}$$

$$E_L^{VFF}(m_V, m_F) = \frac{m_1 m_F^2}{m_V} \times \left[\frac{1}{m_V^2} \left(b_0^{(12)} + b_1^{(1)} - (m_1^2 + m_2^2 - 2m_F^2) C_1 \right) - C_0 + 4C_1 \right], \tag{12}$$

$$E_R^{VFF}(m_V, m_F) = \frac{m_2 m_F^2}{m_V} \times \left[\frac{1}{m_V^2} \left(b_0^{(12)} - b_1^{(2)} + (m_1^2 + m_2^2 - 2m_F^2) C_2 \right) - C_0 - 4C_2 \right], \tag{13}$$

$$E_L^{HFF}(a_1, a_2, v_1, v_2) = \frac{m_1 m_F^2}{v_2} \times \left[\frac{a_1 a_2}{v_1 v_2} b_0^{(12)} + \frac{a_1^2}{v_1^2} m_2^2 (2C_2 + C_0) + \frac{a_2^2}{v_2^2} m_F^2 (C_0 - 2C_1) + \frac{a_1 a_2}{v_1 v_2} \left(2m_2^2 C_2 - (m_1^2 + m_2^2) C_1 + (m_F^2 + m_h^2 + m_2^2) C_0 \right) \right], \tag{14}$$

$$E_R^{HFF}(a_1, a_2, v_1, v_2) = \frac{m_2 m_F^2}{v_2} \times \left[\frac{a_1 a_2}{v_1 v_2} b_0^{(12)} + \frac{a_1^2}{v_1^2} m_1^2 (C_0 - 2C_1) + \frac{a_2^2}{v_2^2} m_F^2 (C_0 + 2C_2) + \frac{a_1 a_2}{v_1 v_2} \left(-2m_1^2 C_1 + (m_1^2 + m_2^2) C_2 + (m_F^2 + m_h^2 + m_1^2) C_0 \right) \right], \tag{15}$$

$$E_L^{FV}(m_F, m_V) = \frac{-m_1 m_2^2}{m_V (m_1^2 - m_2^2)} \left[\left(2 + \frac{m_F^2}{m_V^2} \right) \left(b_1^{(1)} + b_1^{(2)} \right) + \frac{m_1^2 b_1^{(1)} + m_2^2 b_1^{(2)}}{m_V^2} - \frac{2m_F^2}{m_V^2} \left(b_0^{(1)} - b_0^{(2)} \right) \right], \tag{16}$$

$$E_R^{FV}(m_F, m_V) = \frac{m_1}{m_2} E_L^{FV}, \tag{17}$$

$$E_L^{FH}(a_1, a_2, v_1, v_2) = \frac{m_1}{v_1 (m_1^2 - m_2^2)} \left[m_2^2 \left(m_1^2 \frac{a_1^2}{v_1^2} + m_F^2 \frac{a_2^2}{v_2^2} \right) \left(b_1^{(1)} + b_1^{(2)} \right) + m_F^2 \frac{a_1 a_2}{v_1 v_2} \left(2m_2^2 b_0^{(1)} - (m_1^2 + m_2^2) b_0^{(2)} \right) \right], \tag{18}$$

$$E_R^{FH}(a_1,a_2,v_1,v_2)=\frac{m_2}{v_1(m_1^2-m_2^2)}\left[m_1^2\left(m_2^2\frac{a_1^2}{v_1^2}+m_F^2\frac{a_2^2}{v_2^2}\right)\left(b_1^{(1)}+b_1^{(2)}\right)\right.$$
$$\left.+m_F^2\frac{a_1a_2}{v_1v_2}\left(-2m_1^2b_0^{(2)}+(m_1^2+m_2^2)b_0^{(1)}\right)\right]. \quad (19)$$

The notations are introduced as follows. All the b- and C-functions are defined in the Appendix A, where C-functions are well-known as Passarino–Veltman (PV) functions of one-loop three points and b-functions are the finite parts of the two-point functions. For convenience, m_{e_a} and m_{e_b} in the Feynman rules are denoted as m_1, m_2, corresponding to the masses of the final leptons in the LFV decays $h^0\to l_1^- l_2^+$. Other parameters are masses of the neutral Higgs m_{h^0}, and the virtual particles in the loops, including gauge boson mass m_V, charged Higgs mass m_h and fermion masses m_F. Specially, the masses of the virtual fermions are denoted as $m_a\equiv m_F$ for convenience. The parameters a_1, a_2, v_1 and v_2 given in the Feynman rules in the Fig. 2, where v_1, v_2 are VEVs giving masses for normal and exotic leptons/active neutrinos; a_1, a_2 relate the mixing parameters of the charged Higgses coupling with these leptons.

The set of the form factors (4)–(19) was calculated in details in the Appendix B which we find them consistent with calculations using Form [29]. These form factors are simpler than those calculated in the appendix because they contain only terms contributing to the final amplitude of the LFVHD. The excluded terms are come from the two reasons: i) those do not contain the neutral leptons in the loop so they vanish after summing all virtual leptons, reflecting the GIM mechanism; ii) the divergent terms defined by (A.3). The second is true only when the final contribution is assumed to be finite. This is right for the models having no tree level LFV couplings of μ–τ. The 3-3-1LHN model we will consider in this work satisfies this condition and the divergent cancellation is checked precisely in the Appendix B. Another remark is that the divergent term (A.3) contains a conventional choice of $\ln\mu^2/m_h^2$ in which m_h can be replaced by an arbitrary fixed scale. We find that only the contributions of the diagram 1d) and sum of two diagrams 1g) and 1h) are finite.

Now the form factors $\Delta_{L,R}$ can be written as the sum of all $E_{L,R}$ functions. The one loop contributions to the LFV decays such as $\Delta_{L,R}$ are finite without using any renormalization procedure to cancel divergences. In addition, $\Delta_{L,R}$ do not depend on the μ parameter arising from the dimensional regularization method used to derive all above scalar $E_{L,R}$ functions in this work. But in general contributions from the separate diagrams in the Fig. 1 do contain the divergences and therefore the particular finite parts $E_{L,R}$ do depend on μ, so it will be nonsense for computing separate contributions.

Using the Feynman 't Hooft gauge, similar expressions of the LFVHD amplitudes as functions of PV-function were introduced in [12,10]. They were applied for LFVHD in the seesaw models, where there are no new contributions from new physical charged Higgses or new gauge bosons. The contributions in this case correspond to those of only four diagrams a), e) g) and h) in the Fig. 1 of this work. So choosing the unitary gauge is more advantageous for calculating LFVHD predicted by models having complicated particle spectra.

There is another simple analytic expressions given details in [15], updated from previous works [30]. It can be applied for not only SUSY models but also the models predicting new heavy scales including 3-3-1 models. The point is that this treatment uses the C-functions with approximation of zero-external momentums of the two charged leptons, i.e. $p_1^2=p_2^2=0$. Unlike the case of LFV decays of $\tau\to\mu\gamma$, the LFVHD contains a large external momentum of neutral Higgs: $2p_1.p_2\simeq|(p_1\pm p_2)^2|=m_h^2\sim\mathcal{O}([100\text{ GeV}]^2)$, which should be included in the

C-functions, as discussed in the Appendix A. This is consistent with discussion on C-functions given in [31].

3. 3-3-1 model with new neutral lepton

In this section we will review a particular 3-3-1 model used to investigate the LFVHD, namely the 3-3-1LHN [23]. We will keep most of all ingredients shown in Ref. [23], while add two new assumptions: i) in order to appear the LFV effects, we assume that apart from the oscillation of the active neutrinos, there also exists the maximal mixing in the new lepton sector; ii) The Higgs potential satisfies a custodial symmetry shown in [22] to avoid large loop contributions of the Higgses to precision tests such as ρ-parameter and flavor neutral changing currents. More interesting, the latter results a very simple Higgs potential in the sense that many independent Higgs self-couplings are reduced and the squared mass matrix of the neutral Higgses can be solved exactly at the tree level. The following will review the needed ingredients for calculating the LFV decay of $h^0 \to l_i^+ l_j^-$.

3.1. Particle content

- Fermion. In each family, all left-handed leptons are included in the $SU(3)_L$ triplets while right-handed ones are always singlets,

$$L'_a = \begin{pmatrix} \nu'_a \\ e'_a \\ N'_a \end{pmatrix}_L \sim \left(1, 3, -\frac{1}{3}\right), \qquad e'_{aR} \sim (1, 1, -1), \qquad N'_{aR} \sim (1, 1, 0), \tag{20}$$

  where the numbers in the parentheses are the respective representations of the $SU(3)_C$, $SU(2)_L$ and $U(1)_X$ gauge groups. The prime denotes the lepton in the flavor basis. Recall that as one of the assumption in [23], the active neutrinos have no right-handed components and their Majorana masses are generated from the effective dimension-five operators. There is no mixing among active neutrinos and exotic neutral leptons.
- Gauge boson. The $SU(3)_L \times U(1)_X$ includes 8 gauge bosons W^a_μ $(a = 1, 8)$ of the $SU(3)_L$ and the X_μ of the $U(1)_X$, corresponding to eight $SU(3)_L$ generators T^a and a $U(1)_X$ generator T^9. The respective covariant derivative is

$$D_\mu \equiv \partial_\mu - i g_3 W^a_\mu T^a - g_1 T^9 X X_\mu. \tag{21}$$

  Denote the Gell-Mann matrices as λ_a, we have $T^a = \frac{1}{2}\lambda_a, -\frac{1}{2}\lambda_a^T$ or 0 depending on the triplet, antitriplet or singlet representation of the $SU(3)_L$ that T^a acts on. The T^9 is defined as $T^9 = \frac{1}{\sqrt{6}}$ and X is the $U(1)_X$ charge of the field it acts on.
- Higgs. The model includes three Higgs triplets,

$$\rho = \begin{pmatrix} \rho_1^+ \\ \rho^0 \\ \rho_2^+ \end{pmatrix} \sim \left(1, 3, \frac{2}{3}\right), \qquad \eta = \begin{pmatrix} \eta_1^0 \\ \eta^- \\ \eta_2^0 \end{pmatrix}, \; \chi = \begin{pmatrix} \chi_1^0 \\ \chi^- \\ \chi_2^0 \end{pmatrix} \sim \left(1, 3, -\frac{1}{3}\right). \tag{22}$$

As normal, the 3-3-1 model has two breaking steps: $SU(3)_L \times U(1)_X \overset{\langle\chi\rangle}{\overbrace{\rightarrow}} SU(2)_L \times U(1)_Y \overset{\langle\rho\rangle,\langle\eta\rangle}{\overbrace{\rightarrow}} U(1)_Q$, leading to the limit $|\langle\chi\rangle| \gg |\langle\rho\rangle|, |\langle\eta\rangle|$. The non-zero $U(1)_G$ charged field η_2^0 and χ_1^0 have zero vacuum expectation (vev) values: $\langle\eta_2^0\rangle = \langle\chi_1^0\rangle = 0$, i.e.

$$\eta_2^0 \equiv \frac{S_2' + iA_2'}{\sqrt{2}}, \qquad \chi_1^0 \equiv \frac{S_3 + iA_3}{\sqrt{2}}. \tag{23}$$

Others neutral Higgs components can be written as

$$\rho^0 = \frac{1}{\sqrt{2}}(v_1 + S_1 + iA_1), \; \eta_1^0 = \frac{1}{\sqrt{2}}(v_2 + S_2 + iA_2), \; \chi_2^0 = \frac{1}{\sqrt{2}}\left(v_3 + S_3' + iA_3'\right). \tag{24}$$

As shown in Ref. [22], after the first breaking step, the corresponding Higgs potential of the 3-3-1 model should keep a custodial symmetry to avoid large FCNCs as well as the large deviation of ρ-parameter value obtained from experiment. This only involves to the ρ and η Higgs scalars which generate non-zero vevs in the second breaking step. Applying the Higgs potential satisfying the custodial symmetry given in [32], we obtain a Higgs potential of the form,

$$\begin{aligned} \mathcal{V} &= \mu_1^2\left(\rho^\dagger\rho + \eta^\dagger\eta\right) + \mu_2^2\chi^\dagger\chi + \lambda_1\left[\rho^\dagger\rho + \eta^\dagger\eta\right]^2 + \lambda_2\left(\chi^\dagger\chi\right)^2 \\ &+ \lambda_{12}\left(\rho^\dagger\rho + \eta^\dagger\eta\right)\left(\chi^\dagger\chi\right) - \sqrt{2}f\left(\epsilon_{ijk}\rho^i\eta^i\chi^k + \text{h.c.}\right), \end{aligned} \tag{25}$$

where f is assumed to be real. Minimizing this potential leads to $v_1 = v_2$ and two additional conditions,

$$\begin{aligned} &\mu_1^2 + 2\lambda_1 v_1^2 + \frac{1}{2}\lambda_{12}v_3^2 = f v_3, \\ &\mu_2^2 + \lambda_2 v_3^2 + \lambda_{12}v_1^2 = \frac{f v_1^2}{v_3}. \end{aligned} \tag{26}$$

We stress that if the custodial symmetry is kept in this 3-3-1 model, the model automatically satisfies most of the conditions assumed in Ref. [23] for purpose of simplifying or reducing independent parameters in the Higgs potential. For this work, which especially concentrates on the neutral Higgses, the most important consequence is that all of the mass basis of Higgses, including the neutral, can be found exactly without reduction of the number of Higgs multiplets.

In the following, we just pay attention to those used directly in this work, i.e. the mass spectra of leptons, gauge bosons and Higgses. Other parts have been mentioned in [23].

3.2. Mass spectra

3.2.1. Leptons

We use the Yukawa terms shown in [23] for generating masses of charged leptons, active neutrinos and heavy neutral leptons, namely

$$-\mathcal{L}^Y_{\text{lepton}} = y^e_{ab}\overline{L'_a}\rho e'_{bR} + y^N_{ab}\overline{L'_a}\chi N'_{bR} + \frac{y^\nu_{ab}}{\Lambda}\left(\overline{(L'_a)^c}\eta^*\right)\left(\eta^\dagger L'_b\right) + \text{h.c.}, \tag{27}$$

where the notation $(L')^c_a = ((\nu'_{aL})^c,\ (e'_{aL})^c,\ (N'_{aL})^c)^T \equiv (\nu'^c_{aR},\ e'^c_{aR},\ N'^c_{aR})^T$ implies that $\psi^c_R \equiv P_R\psi^c = (\psi_L)^c$ with ψ and $\psi^c \equiv C\overline{\psi}^T$ being the Dirac spinor and its charge conjugation, respectively. The Λ is some high energy scale. Remind that $\psi_L = P_L\psi$, $\psi_R = P_R\psi$ where $P_{R,L} \equiv \frac{1\pm\gamma_5}{2}$ are the right- and left-chiral operators. The corresponding mass terms are

$$-\mathcal{L}^Y_{\text{lepton}} = \left[\frac{y^e_{ab}v_1}{\sqrt{2}}\overline{e'_{aL}}e'_{bR} + \frac{y^N_{ab}v_3}{\sqrt{2}}\overline{N_{aL}}N'_{bR} + \text{h.c.}\right] + \frac{y^\nu_{ab}v_2^2}{2\Lambda}\left[(\overline{\nu'^c_{aR}}\nu'_{bL}) + \text{h.c.}\right]. \quad (28)$$

This means that the active neutrinos are pure Majorana spinors corresponding to the mass matrix $(M_\nu)_{ab} \equiv \frac{y^\nu_{ab}v_2^2}{\Lambda}$. This matrix can be proved to be symmetric [33] (chapter 4), therefore the mass eigenstates can be found by a single rotation expressed by a mixing matrix U that satisfies $U^\dagger M_\nu U = \text{diagonal}(m_{\nu_1},\ m_{\nu_2},\ m_{\nu_3})$, where m_{ν_i} $(i = 1, 2, 3)$ are mass eigenvalues of the active neutrinos.

Now we define transformations between the flavor basis $\{e'_{aL,R},\ \nu'_{aL},\ N'_{aL,R}\}$ and the mass basis $\{e_{aL,R},\ \nu_{aL},\ N_{aL,R}\}$:

$$e'^-_{aL} = e^-_{aL},\ \ e'^-_{aR} = e^-_{aR},\ \nu'_{aL} = U_{ab}\nu_{bL},\ N'_{aL} = V^L_{ab}N_{bL},\quad N'_{aR} = V^R_{ab}N_{bR}, \quad (29)$$

where V^L_{ab}, U^L_{ab} and V^R_{ab} are transformations between flavor and mass bases of leptons. Here unprimed fields denote the mass eigenstates. Remind that $\nu'^c_{aR} = (\nu'_{aL})^c = U_{ab}\nu^c_{aR}$. The four-spinors representing the active neutrinos are $\nu^c_a = \nu_a \equiv (\nu_{aL},\ \nu^c_{aR})^T$, resulting the following equalities: $\nu_{aL} = P_L\nu^c_a = P_L\nu_a$ and $\nu^c_{aR} = P_R\nu^c_a = P_R\nu_a$. The upper bounds of recent experiments for the LFV processes in the normal charged leptons are very suppressed [7], therefore suggest that the two flavor and mass bases of charged leptons should be the same.

The relations between the mass matrices of leptons in two flavor and mass bases are

$$\begin{aligned}
m_{e_a} &= \frac{v_1}{\sqrt{2}}y^e_a, \qquad y^e_{ab} = y^e_a\delta_{ab}, \qquad a, b = 1, 2, 3,\\
\frac{v_2^2}{\Lambda}U^\dagger Y^\nu U &= \text{Diagonal}(m_{\nu_1},\ m_{\nu_2},\ m_{\nu_3}),\\
\frac{v_3}{\sqrt{2}}V^{L\dagger}Y^N V^R &= \text{Diagonal}(m_{N_1},\ m_{N_2},\ m_{N_3}),
\end{aligned} \quad (30)$$

where Y^ν and Y^N are Yukawa matrices defined as $(Y^\nu)_{ab} = y^\nu_{ab}$ and $(Y^N)_{ab} = y^N_{ab}$.

The Yukawa interactions between leptons and Higgses can be written according to the lepton mass eigenstates,

$$\begin{aligned}
-\mathcal{L}^Y_{\text{lepton}} &= \frac{m_{e_b}}{v_1}\sqrt{2}\left[\rho^0_1\bar{e}_bP_Re_b + U^*_{ba}\bar{\nu}_aP_Re_b\rho^+_1 + V^{L*}_{ba}\overline{N}_aP_Re_b\rho^+_2 + \text{h.c.}\right]\\
&+ \frac{m_{N_a}}{v_3}\sqrt{2}\left[\chi^0_2\bar{N}_aP_RN_a + V^L_{ba}\bar{e}_bP_RN_a\chi^- + \text{h.c.}\right]\\
&+ \frac{m_{\nu_a}}{v_2}\left[S_2\overline{\nu_a}P_L\nu_b + \frac{1}{\sqrt{2}}\eta^+\left(U^*_{ba}\overline{\nu_a}P_Le_b + U_{ba}\overline{e^c_b}P_L\nu_a\right) + \text{h.c.}\right],
\end{aligned} \quad (31)$$

where we have used the Marojana property of the active neutrinos: $\nu^c_a = \nu_a$ with $a = 1, 2, 3$. In addition, using the equality $\overline{e^c_b}P_L\nu_a = \overline{\nu_a}P_Le_b$ for this case the term relating with $\eta^\pm$ in the last line of (31) is reduced to $\sqrt{2}\eta^+\overline{\nu_a}P_Le_b$.

3.2.2. Gauge bosons

It is simpler to write the charged gauge bosons in the form of $W^a T^a$ with T^a being the gamma matrices, namely

$$W^a_\mu T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & W^+_\mu & U^0_\mu \\ W^-_\mu & 0 & V^-_\mu \\ U^{0*}_\mu & V^+_\mu & 0 \end{pmatrix}. \tag{32}$$

The masses of these gauge bosons are:

$$m_W^2 = \frac{g^2 v^2}{4}, \qquad m_U^2 = m_V^2 = \frac{g^2}{4}\left(v_3^2 + \frac{v^2}{2}\right), \tag{33}$$

where we have used the relation $v_1 = v_2 = \frac{v}{\sqrt{2}}$ and the matching condition of the W boson mass in 3-3-1 model with that of the SM.

The covariant derivatives of the leptons contain the lepton–lepton–gauge boson couplings, namely

$$\begin{aligned} \mathcal{L}^D_{\text{lepton}} &= i\overline{L'_a}\gamma^\mu D_\mu L'_a \\ &\to \frac{g}{\sqrt{2}}\Big[U^*_{ba}\overline{\nu_a}\gamma^\mu P_L e_b W^+_\mu + U_{ab}\overline{e_b}\gamma^\mu P_L \nu_a W^-_\mu \\ &+ V^{L*}_{ba}\overline{N_a}\gamma^\mu P_L e_b V^+_\mu + V^L_{ab}\overline{e_b}\gamma^\mu P_L N_a V^-_\mu\Big]. \end{aligned} \tag{34}$$

3.2.3. Higgs bosons

- Singly charged Higgses. There are two Goldstone bosons $G^\pm_W$ and $G^\pm_V$ of the respective singly charged gauge bosons $W^\pm$ and $V^\pm$. Two other massive singly charged Higgses have masses

$$m^2_{H_1} = (1+t^2) f v_3, \qquad m^2_{H_2} = 2 f v_3, \tag{35}$$

where $t \equiv \frac{v_1}{v_3} = \frac{v}{v_3\sqrt{2}} = \tan\theta$. Denoting $s_\theta \equiv \sin\theta,\ c_\theta \equiv \cos\theta$, we get some useful relations

$$m_W = \sqrt{2} m_V s_\theta, \qquad v_3 = \frac{2m_V}{g} c_\theta, \qquad v_1 = v_2 = \frac{2m_V}{g} s_\theta. \tag{36}$$

The relation between two flavor and mass bases of the singly Higgses are

$$\begin{pmatrix} \rho_1^\pm \\ \eta^\pm \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} G^\pm_W \\ H^\pm_2 \end{pmatrix}, \qquad \begin{pmatrix} \rho_2^\pm \\ \chi^\pm \end{pmatrix} = \begin{pmatrix} -s_\theta & c_\theta \\ c_\theta & s_\theta \end{pmatrix}\begin{pmatrix} G^\pm_V \\ H^\pm_1 \end{pmatrix}. \tag{37}$$

- CP-odd neutral Higgses. There are three Goldstone bosons G_Z, $G_{Z'}$ and G'_{U^0}, and two massive CP-odd neutral Higgses H_{A_1} and H_{A_2} with the values of squared masses are

$$m^2_{A_1} = m^2_{H_1} = \frac{(1+t^2)}{2} m^2_{H_2},\ m^2_{A_2} = \frac{(2+t^2)}{2} m^2_{H_2}. \tag{38}$$

The relations between the two bases are:

$$\begin{pmatrix} A_3 \\ A'_2 \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} G_3 \\ H_{A_2} \end{pmatrix}, \quad \begin{pmatrix} A_1 \\ A'_3 \\ A_2 \end{pmatrix} \begin{pmatrix} -s_\theta & \frac{-c_\theta^2}{\sqrt{c_\theta^2+1}} & \frac{c_\theta}{\sqrt{c_\theta^2+1}} \\ c_\theta & \frac{-s_\theta c_\theta}{\sqrt{c_\theta^2+1}} & \frac{s_\theta}{\sqrt{c_\theta^2+1}} \\ 0 & \frac{1}{\sqrt{c_\theta^2+1}} & \frac{c_\theta}{\sqrt{c_\theta^2+1}} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ H_{A_1} \end{pmatrix}. \quad (39)$$

- CP-even neutral Higgses. Apart from the three exactly massive Higgses shown in the Ref. [22], the model predicts one more Goldstone boson G_U and another massive Higgs. The masses and egeinstates of these Higgses are

$$m^2_{h_1^0} = \frac{v_3^2}{2}\left[4\lambda_1 t^2 + 2\lambda_2 + \frac{t^2 f}{v_3} - \sqrt{\Delta}\right],$$
$$m^2_{h_2^0} = \frac{v_3^2}{2}\left[4\lambda_1 t^2 + 2\lambda_2 + \frac{t^2 f}{v_3} + \sqrt{\Delta}\right],$$
$$m^2_{h_3^0} = m^2_{H_1^\pm}, \; m^2_{h_4^0} = m^2_{A_2}, \quad (40)$$

where $\Delta = \left(4\lambda_1 t^2 - 2\lambda_2 - \frac{t^2 f}{v_3}\right)^2 + 8t^2\left(\lambda_{12} - \frac{f}{v_3}\right)^2$. The transformations among the flavor and the mass bases are

$$\begin{pmatrix} S'_2 \\ S_3 \end{pmatrix} \begin{pmatrix} -s_\theta & c_\theta \\ c_\theta & s_\theta \end{pmatrix} = \begin{pmatrix} G'_U \\ h_4^0 \end{pmatrix}, \quad \begin{pmatrix} S_2 \\ S_1 \\ S'_3 \end{pmatrix} = \begin{pmatrix} \frac{-c_\alpha}{\sqrt{2}} & \frac{s_\alpha}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-c_\alpha}{\sqrt{2}} & \frac{s_\alpha}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ s_\alpha & c_\alpha & 0 \end{pmatrix} \begin{pmatrix} h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix}, \quad (41)$$

where $s_\alpha = \sin\alpha$, $c_\alpha = \cos\alpha$ defining by

$$s_\alpha = \frac{4\lambda_1 t^2 - m^2_{h_1^0}/v_3^2}{\sqrt{2\,(2\lambda_1 - f/v_3)^2\, t^2 + \left(4\lambda_1 t^2 - m^2_{h_1^0}/v_3^2\right)^2}},$$
$$c_\alpha = \frac{\sqrt{2}\,(2\lambda_1 - f/v_3)\, t}{\sqrt{2\,(2\lambda_1 - f/v_3)^2\, t^2 + \left(4\lambda_1 t^2 - m^2_{h_1^0}/v_3^2\right)^2}}. \quad (42)$$

In the limit $t \ll 1$ the expression of the lightest neutral even-CP Higgs is

$$m^2_{h_1^0} \simeq v_1^2\left[4\lambda_1 - \frac{(\lambda_{12} - f/v_3)^2}{\lambda_2}\right],$$

where both λ_1 and λ_2 must be positive to guarantee the vacuum stability of the potential (25). This Higgs is easily identified with the SM-like Higgs observed by LHC.

3.3. Couplings for LFV decay of the SM-like Higgs and the amplitude

From the detailed discussions on the particle content of the 3-3-1LHN, the couplings of SM-like Higgs needed for calculating LFVHD are collected in the Table 1.

Table 1
Couplings relating with LFV of SM-like Higgs decays in the 3-3-1LHN model, where $\lambda_{h^0 H_1 H_1} = s_\alpha c_\theta^2 \lambda_{12} + 2 s_\theta^2 \lambda_2 - \sqrt{2}(2 c_\alpha c_\theta^2 \lambda_1 + s_\theta^2 \lambda_{12}) t_\theta - c_\theta s_\theta \frac{f}{v_3}\sqrt{2}$. Here we only consider the couplings the unitary gauge.

Vertex	Coupling	Vertex	Coupling
$\bar{N}_a e_b H_1^+$	$-i\sqrt{2} V_{ba}^{L*}\left(\frac{m_{e_b}}{v_1} c_\theta P_R + \frac{m_{N_a}}{v_3} s_\theta P_L\right)$	$\bar{e}_a N_b H_1^-$	$-i\sqrt{2} V_{ba}^{L}\left(\frac{m_{e_b}}{v_1} c_\theta P_L + \frac{m_{N_a}}{v_3} s_\theta P_R\right)$
$\bar{\nu}_a e_b H_2^+$	$-i U_{ba}^{L*}\left(\frac{m_{e_b}}{v_1} P_R + \frac{m_{\nu_a}}{v_2} P_L\right)$	$\bar{e}_b \nu_a H_2^-$	$-i U_{ab}^{L}\left(\frac{m_{e_b}}{v_1} P_L + \frac{m_{\nu_a}}{v_2} P_R\right)$
$\bar{N}_a N_a h_1^0$	$\frac{-i m_{N_a} s_\alpha}{v_3}$	$\bar{e}_a e_a h_1^0$	$\frac{i m_{e_a}}{v_1}\frac{c_\alpha}{\sqrt{2}}$
$\bar{N}_a e_b V_\mu^+$	$\frac{ig}{\sqrt{2}} V_{ba}^{L*}\gamma^\mu P_L$	$\bar{e}_b N_a V_\mu^-$	$\frac{ig}{\sqrt{2}} V_{ab}^{L}\gamma^\mu P_L$
$\bar{\nu}_a e_b W_\mu^+$	$\frac{ig}{\sqrt{2}} U_{ba}^{L*}\gamma^\mu P_L$	$\bar{e}_b \nu_a W_\mu^-$	$\frac{ig}{\sqrt{2}} U_{ab}^{L}\gamma^\mu P_L$
$W^{\mu+} W_\mu^- h_1^0$	$-i g m_W c_\alpha$	$V^{\mu+} V_\mu^- h_1^0$	$\frac{i g m_V}{\sqrt{2}}(\sqrt{2} s_\alpha c_\theta - c_\alpha s_\theta)$
$h_1^0 H_1^+ V^{\mu-}$	$\frac{ig}{2\sqrt{2}}(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta)(p_{h_1^0} - p_{H_1^+})_\mu$	$h_1^0 H_1^- V^{\mu+}$	$\frac{ig}{2\sqrt{2}}(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta)(p_{H_1^-} - p_{h_1^0})_\mu$
$h_1^0 H_1^+ H_1^-$	$-i v_3 \lambda_{h^0 H_1 H_1}$	$h_1^0 H_2^+ H_2^-$	$-i v_1\left[-2\sqrt{2} c_\alpha \lambda_1 + \frac{s_\alpha v_3 \lambda_{12} + s_\alpha f}{v_1}\right]$
$\bar{\nu}_a \nu_a h_1^0$	$\frac{i m_{\nu_a}}{v_2}\frac{c_\alpha}{\sqrt{2}}$	$h_1^0 H_2^\pm W_\mu^\pm$	0

Matching the Feynman rules in the Fig. 2, we have the specific relations among the vertex parameters and the couplings in the 3-3-1LHN, namely for the exotic leptons

$$a_1 \to c_\theta,\ a_2 \to a_3 = s_\theta,\ v_1 = \frac{2m_V}{g} s_\theta,\ v_2 \to v_3 = \frac{2m_V}{g} c_\theta,$$
$$\frac{a_1}{v_1} = \frac{g}{2m_V}\frac{c_\theta}{s_\theta},\ \frac{a_3}{v_3} = \frac{g}{2m_V}\frac{s_\theta}{c_\theta},\ \frac{a_1 a_3}{v_1 v_3} = \frac{g^2}{4m_V^2}, \tag{43}$$

and the active neutrinos,

$$a_1, a_2 \to 1, \qquad v_1, v_2 \to v_1 = v_2 = \frac{v}{\sqrt{2}} = \frac{\sqrt{2} m_W}{g}, \qquad \frac{a_1}{v_1} = \frac{a_2}{v_2} = \frac{g}{\sqrt{2} m_W}. \tag{44}$$

The expression of Δ_L is separated into two parts, namely

$$\begin{aligned}
\Delta_L^N = \sum_a V_{1a}^L V_{2a}^{L*}\frac{1}{64\pi^2\sqrt{2}}\Big[\, & 2g^3\left(-c_\alpha s_\theta + \sqrt{2} s_\alpha c_\theta\right) \times E_L^{FVV}(m_{N_a}, m_V) \\
& + (-2g^2)\left(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta\right) \times E_L^{FVH}(a_1, a_3, v_1, v_3, m_{N_a}, m_V, m_{H_1^\pm}) \\
& + (-2g^2)\left(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta\right) \times E_L^{FHV}(a_1, a_3, v_1, v_3, m_{N_a}, m_V, m_{H_1^\pm}) \\
& + \left(-4\sqrt{2}\lambda_{h^0 H_1 H_1}\right) \times E_L^{FHH}(a_1, a_2, v_1, v_2, m_{\nu_a}, m_{H_2^\pm}) \\
& + \frac{g^3 s_\alpha \sqrt{2}}{c_\theta} \times E_L^{VFF}(m_V, m_{\nu_a}) \\
& + \left(-8\sqrt{2} s_\alpha\right) E_L^{HFF}(a_1, a_3, v_1, v_3, m_{\nu_a}, m_{H_1^\pm})
\end{aligned}$$

$$+ \frac{-g^3 c_\alpha}{s_\theta} \times E_L^{FV}(m_V, m_{N_a})$$
$$+ 8c_\alpha \times E_L^{FH}(a_1, a_3, v_1, v_3, m_{N_a}, m_{H_1^\pm})\Big] \quad (45)$$

from neutral exotic leptons and

$$\Delta_L^\nu = \sum_a U_{1a} U_{2a}^* \frac{1}{64\pi^2} \Big[\left(-2g^3 c_\alpha\right) E_L^{FVV}(m_{\nu_a}, m_W)$$
$$+ (-4\lambda_{h^0 H_2 H_2}) \times E_L^{FHH}(a_1, a_2, v_1, v_2, m_{\nu_a}, m_{H_2^\pm})$$
$$+ \left(-g^3 c_\alpha\right) E_L^{VFF}(m_W, m_{\nu_a})$$
$$+ (2\sqrt{2} c_\alpha) E_L^{HFF}(a_1, a_2, v_1, v_2, m_{\nu_a}, m_{H_2^\pm})$$
$$+ \left(-g^3 c_\alpha\right) E_L^{FV}(m_V, m_{\nu_a})$$
$$+ (2\sqrt{2} c_\alpha) E_L^{FH}(a_1, a_2, v_1, v_2, m_{\nu_a}, m_{H_2^\pm})\Big]. \quad (46)$$

Similarly for the Δ_R we have

$$\Delta_R^N = \Delta_L^N (E_L \to E_R), \qquad \Delta_R^\nu = \Delta_L^\nu (E_L \to E_R). \quad (47)$$

Before going to the numerical calculation we remind that the divergent cancellations in two separate sectors of neutrinos and exotic leptons are presented precisely in the second subsection of the Appendix B.

4. Numerical investigation

4.1. Setup parameters

In the model under consideration, the new parameters we pay attention to are the $SU(3)_L$ scale v_3, the mass of the lightest active neutrino, masses of the three neutral heavy leptons, Higgs masses and mixing parameters of leptons and Higgses. The Higgs part relates with the Higgs self-couplings in the scalar potential: λ_1, λ_2, λ_{12} and f. The first two free parameters we choose are the v_3 and mass of the H_2 given in (35). Then the f parameter can be determined by

$$f = \frac{m_{H_2}^2}{2v_3}. \quad (48)$$

Another parameter that can be fixed is the mass of the neutral SM-like Higgs [5] with the value of about $m_{h_1^0} = 125.1$ GeV. Note that two Higgs masses $m_{h_1^0}^2$ and $m_{h_2^0}^2$ shown in (40) are roots of the equation $x^2 + ax + b = 0$, where $-a = m_{h_1^0}^2 + m_{h_2^0}^2 = v_3^2\left(4\lambda_1 t^2 + 2\lambda_2 + t^2 f/v_3\right)$ and $b = m_{h_1^0}^2 m_{h_2^0}^2 = 2v_1^2 v_3^2 \left[2\lambda_1 \times \left(2\lambda_2 + t^2 f/v_3\right) - (\lambda_{12} - f/v_3)^2\right]$. This means that $m_{h_1^0}^4 + a \times m_{h_1^0}^2 + b = 0$, giving a relation among λ_2, λ_1 and λ_{12}:

$$\lambda_2 = \frac{t_\theta^2}{2}\left(\frac{m_{h_1^0}^2}{v_1^2} - \frac{m_{H_2}^2}{2v_3^2}\right) - \frac{\left(\lambda_{12} - m_{H_2}^2/2v_3^2\right)^2}{-4\lambda_1 + m_{h_1^0}^2/v_1^2}.$$

Because the λ_1, λ_2 and λ_{12} are factors of quartic terms in the Higgs potential (25), they must satisfy the unbounded from below (UFB) conditions that guarantee the stability of the vacuums of the considering model. According to the Ref. [42], these conditions are easily found as follows. Defining $\rho^\dagger\rho + \eta^\dagger\eta = h_1^2$ and $\chi^\dagger\chi = h_2^2$, the quartic part of the Higgs potential (25) has form of $V_4 = \lambda_1(h_1^2)^2 + \lambda_{12}h_1^2h_2^2 + \lambda_2(h_2^2)^2$. In the basis (h_1^2, h_2^2) the V_4 corresponds to the 2×2 matrix that must satisfy the conditionally positive conditions as follows:

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \text{and } \frac{\lambda_{12}}{2} + \sqrt{\lambda_1\lambda_2} \geq 0. \tag{49}$$

In our calculation, apart from positive λ_1 and λ_2 we will choose $\lambda_{12} > 0$ so that all conditions given in (49) are always satisfied.

To identify h_1^0 with the SM Higgs, the h_1^0 must satisfy new constrains from LHC, as discussed in [43]. Namely, the mixing angle α of neutral Higgses, defined in (42), should be constrained from the $h_1^0W^+W^-$ coupling. Following [43] the we can identify that $-c_\alpha \equiv 1 + \epsilon_W$ where $\epsilon_W = -0.15 \pm 0.14$ is the universal fit for the SM Higgs. This results the constraint of c_α as

$$-0.99 \leq c_\alpha \leq -0.71. \tag{50}$$

By canceling a factor of t in (42), we have a simpler expression

$$c_\alpha = \frac{\sqrt{2}\left(2\lambda_1 - \frac{m_{H_2}^2}{2v_3^2}\right)}{\sqrt{2\left(2\lambda_1 - \frac{m_{H_2}^2}{2v_3^2}\right)^2 + t^2\left(4\lambda_1 - m_{h_1^0}^2/v_1^2\right)^2}},$$

which shows that $c_\alpha < 0$ when $m_{H_2} > 2v_3\sqrt{\lambda_1}$ and $c_\alpha \to -1$ when $t \ll 1$. The lower constraint of c_α in (50) gives a very interesting relation among λ_1, v_3 and m_{H_2}, namely $m_{H_2}^2$ can be written as

$$m_{H_2}^2 = v_3^2\left[4\lambda_1 + \left|4\lambda_1 - \frac{m_{h_1^0}^2}{v_1^2}\right| \times \frac{\sqrt{2}|c_\alpha|}{\sqrt{1-c_\alpha^2}} \times \frac{v_1}{v_3}\right]. \tag{51}$$

If the lower constraint in (50) is not considered, $m_{H_2}^2$ can be arbitrary large when $|c_\alpha| \to 1$. In contrast, the constraint (50) gives a consequence $\frac{\sqrt{2}|c_\alpha|}{\sqrt{1-c_\alpha^2}} \sim \mathcal{O}(1)$. Combining with $m_{h_1^0}^2/v_1^2 \simeq 0.52$, we obtain a rather strict relation $m_{H_2} \simeq 2v_3\sqrt{\lambda_1}$ if $v_3 \gg v_1 \simeq 246/\sqrt{2}$ GeV and λ_1 is large enough. On the other hand, this relation will not hold if the custodial symmetry assumed in the Higgs potential (25) is only an approximation. Hence in the numerical calculation, for the general case we will first investigate the LFVHD without the constraint (50). This constraint will be discussed in the final.

Regarding to the parameters of active neutrinos we use the recent results of experiment. In particularly, if the mixing parameters in the active neutrino sector are parameterized by

$$U(\theta_{12},\theta_{13},\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{23} & \sin\theta_{23} \\ 0 & -\sin\theta_{23} & \cos\theta_{23} \end{pmatrix}\begin{pmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{pmatrix} \times \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} & 0 \\ -\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{52}$$

Because U^L has a small deviation from the well-known neutrino mixing matrix U^{MNPS} so we ignore this deviation [34]. We will use the best-fit values of neutrinos oscillation parameters given in [35],

$$\begin{aligned} &\Delta m_{21}^2 = 7.60 \times 10^{-5}\ \mathrm{eV}^2, \quad \Delta m_{31}^2 = 2.48 \times 10^{-3}\ \mathrm{eV}^2, \\ &\sin^2\theta_{12} = 0.323,\ \sin^2\theta_{23} = 0.467,\ \sin^2\theta_{13} = 0.0234, \end{aligned} \tag{53}$$

and mass of the lightest neutrino will be chosen in range $10^{-6} \le m_{\nu_1} \le 10^{-1}$ eV, or $10^{-15} \le m_{\nu_1} \le 10^{-10}$ GeV. This range satisfies the condition $\sum_b m_{\nu_b} \le 0.5$ eV obtained from the cosmological observable. The remain two neutrino masses are $m_{\nu_b}^2 = m_{\nu_1}^2 + \Delta m_{\nu_b 1}^2$. We note that the above case corresponds to the normal hierarchy of active neutrino masses. In the 3-3-1LHN, the inverted case gives the same result so we do not consider here.

The mixing matrix of the exotic leptons is also parameterized according to (52). In particularly it is unknown and defined as $V^L \equiv U^L(\theta_{12}^N, \theta_{13}^N, \theta_{23}^N)$. If all $\theta_{ij}^N = 0$, all contributions from exotic leptons to $\Delta_{L,R}$ will be exactly zero. In the numerical computation, we consider only the cases of maximal mixing in the exotic lepton sector, i.e. each θ_{ij}^N gets only the value of $\pi/4$ or zero. There are three interesting cases: i) $\theta_{12}^N = \pi/4$ and $\theta_{13}^N = \theta_{23}^N = 0$; ii) $\theta_{12}^N = \theta_{13}^N = \theta_{23}^N = \pi/4$; and iii) $\theta_{12}^N = \theta_{13}^N = \pi/4$ and $\theta_{23}^N = -\pi/4$. The other cases just change minus signs in the total amplitudes, and do not change the final results of LFVHD branching ratios. For example the mixing matrix of first case is

$$V^L = U(\pi/4, 0, 0) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{54}$$

Our numerical investigation will pay attention to the first case, where the third exotic lepton does not contribute to the LFVHD decays. The two other cases are easily deduced from this investigation.

From the above discussion, we chose the following unknown parameters as free parameters: v_3, m_{H_2}, λ_1, λ_{12}, m_{ν_1} and m_{N_a} $(a = 1, 2, 3)$. The vacuum stability of the potential (25) results the consequence $\lambda_{1,2} > 0$. In order to be consistent with the perturbativity property of the theory, we will choose $\lambda_1, |\lambda_{12}| < \mathcal{O}(1)$. The numerical check shows that the LFVHD branching ratio depends weakly on the changes of these Higgs self-couplings in this range. Therefore we will fix $\lambda_1 = \lambda_{12} = 1$ without loss of generality. These values of λ_1 and λ_{12} also satisfy all UFB conditions (49). In addition, the Yukawa couplings in the Yukawa term (27) should have a certain upper bound, for example in order to be consistent with the perturbative unitarity limit [36]. Because the vev v_3 generates masses for exotic leptons from the Yukawa interactions (28), following [10] we assume the upper bound of the lepton masses as follows

$$\left|\frac{m_{N_a}}{v_3}\right|^2 \le \left|\frac{y_{ij}^N}{\sqrt{2}}\right|^2 < 3\pi. \tag{55}$$

After investigating the dependence of the LFVHD on the Yukawa couplings through the ratio $\frac{m_{N_a}}{v_3}$ we will fixed $m_{N_2}/v_3 = 0.7$ and 2 corresponding to the two cases of lower and larger than 1 of the Yukawa couplings.

Unlike the assumption in [23] where $f = v_3/2$, we treat f as a free parameter relating with m_{H_2} by the equation (48), so the condition of candidates of DM may be changed. We stress that the correlation between m_{H_2} and v_3 is very important to get maximal values of LFVHD branching ratio. The singly charged Higgs bosons have been being searched at LHC, namely the decays $H^+ \to c\bar{s}$ or $H^\pm \to WZ$ with ATLAS [37], and decays to fermions with CMS [38]. The ATLAS gives a lower bound of 1 TeV while that from CMS is about 600 GeV. But in the 3-3-1LHN model, there is no coupling $H_1^\pm W^\mp Z$, while the coupling $H_2^\pm W^\mp Z$ is extremely small when $v_1 = v_2$. In addition, only the H_2 decay has been searched by CMS so the lower bound of $m_{H_2} \geq 600$ GeV should be applied. The value of m_{H_2} should also satisfy $\frac{m_{H_2}}{v_3} \leq \mathcal{O}(1)$, resulting an upper bound depending on the $SU(3)_L$ scale.

The value of v_3 should be consistent with the lower bound of Z' mass from experimental searches [39], addressing directly for 3-3-1 models [19,40], where $m_{Z'}$ must be above 2.5 TeV. It is enough using an approximate relation of $m_{Z'}$ and v_3: $m^2_{Z'} \simeq g^2 v_3^2 c_W^2/(3-4s_W^2)$ where $s_W = \sin\theta_W$ and $c_W = \cos\theta_W$ with θ_W being the Weinberg angle. Then v_3 should be above 6 TeV. For understanding the qualitative properties of the LFVHD, our investigation will pay attention on the range of 4 TeV $< v_3 <$ 10 TeV.

To see the correlation between singly charged Higgses, the neutral leptons and the v_3, the range of m_{H_2} will be chosen as 0.5 TeV $< m_{H_2} <$ 20 TeV. The default value of $m_{N_1} = 2$ TeV is used. The value of m_{N_2} is chosen later.

The other well-known parameters are fixed [41]: W boson mass $m_W = 80.385$ GeV, the weak-mixing angle value $s^2_W = 0.231$, the fine-structure constant at the electroweak scale $\alpha = e^2/4\pi = 1/128$, the total decay width of the SM Higgs $\Gamma_H \simeq 4.07$ GeV. The mass of this Higgs is fixed as $m_H = 125.09$ GeV. These two values are assumed to be the total decay width and mass of the SM-like Higgs considered in this work.

A main point that can distinguish the LFVHD characteristics in the 3-3-1 models with the other well-known models beyond SM, including the seesaw and SUSY models, is the relation of new neutral lepton masses and the Yukawa couplings which directly relate to the LFVHD. In particular, because all neutral leptons in 3-3-1LHN receive masses from the Yukawa terms, so their masses must be bounded from above because of the inequality (55) and a similar one for active neutrinos. This also implies that maximal values of exotic lepton masses depend on the $SU(3)_L$ scale v_3. While in the seesaw models with new singlets right-handed neutrinos, the mass terms of sterile neutrinos are mainly come from the private Majorana mass terms and no new Yukawa couplings appear. So the mass ranges of new sterile neutrinos may be very wide, even if their effects to the Yukawa couplings of the active neutrinos are included [10]. Similar, in the SUSY models, the appearance of the soft terms leads to the consequence that masses of new superpartners affecting to LFVHD are mainly come from these soft terms. In conclusion, the study of LFVHD in 3-3-1LHN can give some interesting information on Yukawa couplings of exotic leptons and the $SU(3)_L$ scale v_3.

4.2. Numerical result

If the mixing parameters among all exotic leptons are zero or all of their masses are degenerate, then the contributions to the LFVHD of these exotic leptons are zero, too. Then branching ratio of the LFVHD $h_1^0 \to \mu\tau$ depends on only active neutrino sector, in which the mixing parameters as well as masses are almost known. The numerical results in this case are shown in the Fig. 3. The LFVHD does not depend on the value of the lightest active neutrino, but increases very slightly with the increasing of v_3 and m_{H_2}. Because both values of v_3 and m_{H_2} are in the TeV scale, the contribution of the active neutrinos is extremely small compared with the recent experimental sensitivity, so we can neglect it in the next calculation.

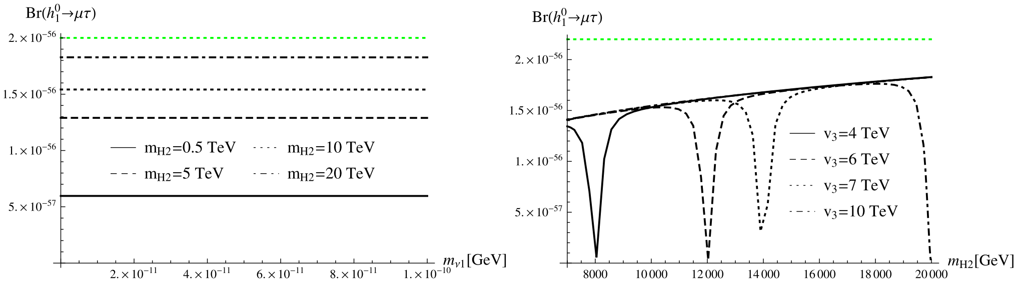

Fig. 3. Branching ratio of LFVHD as function of m_{ν_1} (left panel) or m_{H_2} (right panel) where contributions are come from only active neutrinos in the loops.

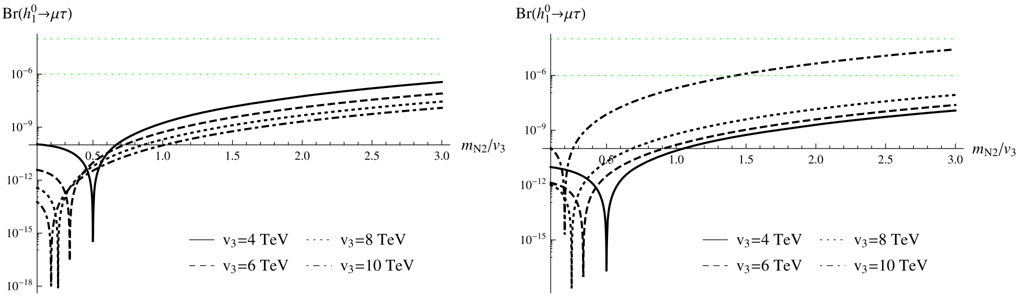

Fig. 4. Branching ratio of LFVHD as function of m_{N_2}/v_3, which is proportional to Yukawa couplings of exotic leptons, $m_{H_2} = 2$ (20) TeV in the left (right) panel. The upper green lines correspond to the value of 10^{-4}. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Now we begin considering the contribution of exotic leptons. Firstly the dependence of the branching ratio of LFVHD on the Yukawa couplings, or the ratio of m_{N_2}/v_3, is shown in the Fig. 4. The branching ratio enhances rapidly with the increasing of the Yukawa couplings. In addition, the branching ratio is small, below 10^{-6}, with small $m_{H_2} = 2$ TeV, and rather large with larger m_{H_2}. In particular for $m_{H_2} = 20$ TeV, the branching ratio can reach 10^{-5}. Both of the largest values in the two panels correspond to the largest values of the Yukawa couplings. The deep wells show the zero values of the LFVHD branching ratio when the two exotic lepton masses are exactly degenerate at the default value of $m_{N_1} = 2$ TeV. For the small value of m_{H_2}, the small v_3 (the black line in the left panel) gives larger BR($h_1^0 \to \mu\tau$). In contrast, the larger values of m_{H_2} and v_3 (the dot-dash line in the right panel) give large BR($h_1^0 \to \mu\tau$). The one more interesting property is that the branching ratio seems to be unchanged with very small values of m_{N_2}, implies that the small exotic lepton masses give small contribution the to LFVHD. The constant values of LFVHD in the right-hands sides of the wells are from the contributions of $m_{N_1} = 2$ TeV when m_{N_2} is much smaller than m_{N_1}.

For qualitative estimation, we have checked $\Delta_{L,R}$ as functions of mass parameters as follows. We divide them into two parts: $\Delta_{L,R} = f(m_H, v_3, m_{N_a}) + g(m_H, v_3, m_{N_a}) \times \ln(m_{N_a}^2)$ and consider their behavior when one of the parameters approaches zero or infinity. Note that the logarithm factors are very important because they can give very large contributions even with the very small values of m_{N_a}. For the exotic lepton masses, there are two interesting properties:

$$\lim_{m_{N_a} \to 0} g(m_H, v_3, m_{N_a}) \ln(m_{N_a}^2) = 0 \text{ and } \lim_{m_{N_a} \to \infty} g(m_H, v_3, m_{N_a}) \ln(m_{N_a}^2) = \pm\infty, \quad (56)$$

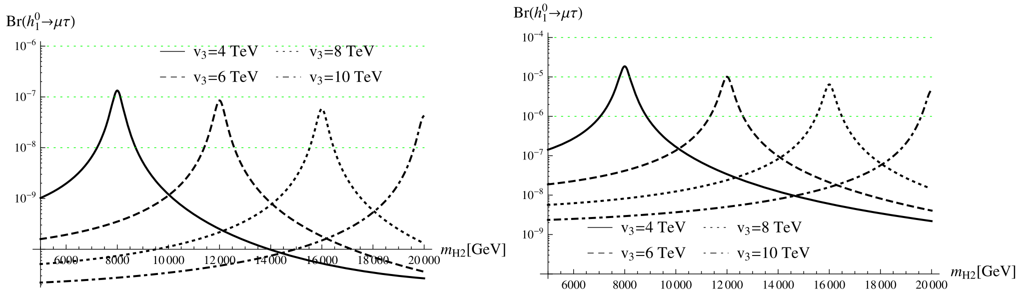

Fig. 5. Branching ratio of LFVHD as function of m_{H_2}, $m_{N_2}/v_3 = 0.7$ (2) in the left (right) panel.

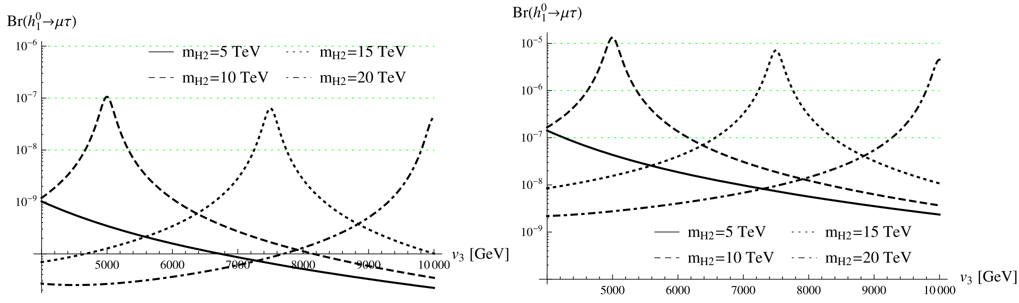

Fig. 6. Branching ratio of LFVHD as function of v_3, $m_{N_2}/v_3 = 0.7$ (2) in the left (right) panel.

with the assumption that all other parameters are fixed and the exotic lepton masses do not have any upper bounds. The first limitation explains why small exotic leptons give suppressed contributions to LFVHD. If the upper bound of the Yukawa couplings, namely (55), is applied, the value of the second limitation in (56) becomes zero. In the well-known classes of models such as the models with singlet right-handed neutrinos or the SUSY models, the upper bounds of new lepton masses or superpartner masses do not relate with the vevs of Higgses, because these masses are also come from other sources as the singlet mass terms or the soft terms. So the Br($h_1^0 \to \mu\tau$) increases with increasing of the new mass scales [10]. Hence the upper bound of the LFVHD will result to the upper bound of these new mass scales. In contrast, in the frame work of the 3-3-1 models, the LFVHD will give much of important information of the Yukawa couplings of the exotic leptons.

As showed in the Fig. 4, the Br($h_1^0 \to \mu\tau$) depends clearly on m_{N_2}/v_3 whether this ratio is larger or smaller than 1. From now we will consider two fixed values of $m_{N_2}/v_3 = 0.7$ and 2, without any inconsistence in the final results.

The Fig. 5 shows the dependence of LFVHD on the mass of m_{H_2}. The first property we can see is that the LFVHD branching ratio always has an upper bound that decreases with increasing v_3. In other word, it has an maximal value depending strictly on the constructive correlation of v_3 and m_{H_2}. But if the Yukawa couplings are small, this maximum seems never reach the value of 10^{-6}. The case of the large Yukawa couplings is more interesting because maximal LFVHD can be asymptotic 10^{-5}, provided that v_3 is small enough, see the right panel.

The effects of v_3 on LFVHD are shown in the Fig. 6. Again we can see that the maximal values can reach 10^{-7} and 10^{-5} for respective small and large Yukawa couplings.

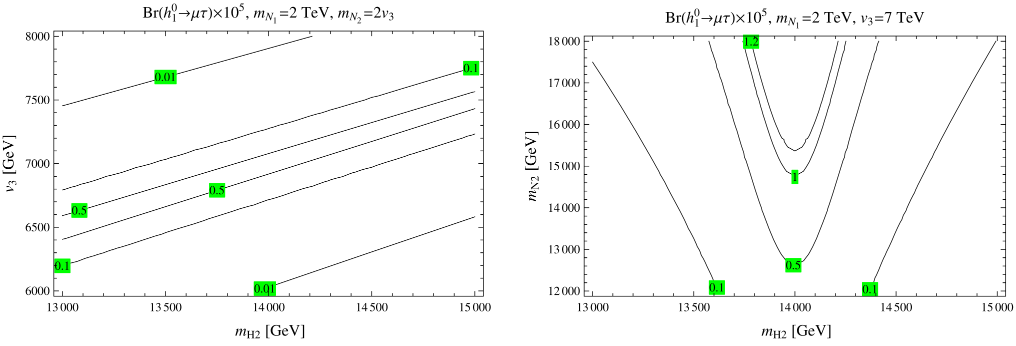

Fig. 7. Contour plots of LFVHD as function of v_3 and m_{H_2} in the left (right) panel.

Combining both Figs. 5 and 6, we conclude that the construction correlation of m_{H_2} and v_3 is the necessary condition for maximal peaks and the appearance of vertices are independent with Yukawa couplings. But the maximal values of LFVHD branching ratio depend directly on the amplitudes of the Yukawa couplings and can reach 10^{-5}.

The Fig. 7 represents some particular regions of the parameter space to get the large values of LFVHD Br($h_1^0 \to \mu\tau$). Especially the values larger than 10^{-5} are the maximal values of LFVHD that the 3-3-1LHN can predict when the lower bound of v_3 is 6 TeV. In addition, the left panel shows the case of $m_{N_2}/v_3 = 2$, the parameters satisfying Br($h_1^0 \to \mu\tau$) $\geq 0.5 \times 10^{-6}$ is very narrow, implies a very strict relation of v_3 and m_{H_2} if this large amount of the branching ratio is observed. The right panel shows the dependence of Br($h_1^0 \to \mu\tau$) on the Yukawa couplings and m_{H_2} with $v_3 = 7$ TeV. Clearly, the maximal peak of LFVHD corresponds to $m_{H_2} \simeq 14$ TeV and does not depend on the Yukawa couplings. But the maximal values do, in this case Br($h_1^0 \to \mu\tau$) $\geq 0.5 \times 10^{-5}$ if only $m_{N_2} \geq 14.5$ TeV. Furthermore, the region having Br($h_1^0 \to \mu\tau$) $\geq 0.5 \times 10^{-5}$ opens wider with larger Yukawa couplings.

Finally, we should pay attention to the case satisfying the constraint of universal Higgs fit (50). In the above numerical investigation, we have fixed $\lambda_1 = 1$, which corresponds to $m_{H_2} \simeq 2v_3\sqrt{\lambda_1} = 2v_3$ satisfying the constraint. It is very interesting that all maximal peaks of LFVHD appearing in the numerical calculations correspond to this relation among m_{H_2}, v_3 and λ_1. Therefore the universal Higgs fit confirms more strongly that the 3-3-1LHN predicts the large branching ratios of LFVHD.

5. Conclusion

For studying the LFVHD in the 3-3-1LHN model, we have introduced form factors expressing the one-loop contributions corresponding to relevant Feynman diagrams in the unitary gauge. We have checked that the total contribution is finite, all of the divergences appearing in particular diagrams cancel among one to another. Although the above form factors are calculated for the 3-3-1LHN, they can be applied for other 3-3-1 models and in general for many other models beyond the SM with the same class of particles. In numerical investigation the LFVHD in the case of maximal mixing between the first two exotic neutral leptons, we find that the branching ratio Br($h_1^0 \to \mu\tau$) depends the mostly on Yukawa couplings of neutral exotic leptons and the $SU(3)_L$ scale v_3. For small $y_{ij}^N \simeq 1$, equivalently $m_{N_2}/v_3 \simeq 0.7$, this branching ratio is always lower than 10^{-6}, and even that of about 10^{-7}, the parameter space is very narrow. In contrast, with large Yukawa couplings, for example $y^N_{ij} \simeq 2\sqrt{2}$ or $m_{N_2}/v_3 \simeq 2$, the largest LFVHD branching ratio can reach 10^{-5} and does not depend on the small values of m_{N_1}. These largest values do also depend on the charged Higgs masses and the v_3, thought these seem not as strongly as the Yukawa couplings. The values above 10^{-5} can be found in large region of parameter space with small v_3. With the large v_3, this region is very small, implying some strict relation between parameters of exotic lepton masses, charged Higgs masses and the $SU(3)_L$ scale v_3. The relation arises from the present of both the custodial symmetry in the Higgs potential and the constraint from the universal fit of the Higgs property observed by LHC. This will give interesting information of the 3-3-1LHN model if the LFVHD branching ratio is discovered by experiments at the value of 10^{-5} or larger. Our calculation also indicates that only 3-3-1 models with new heavy leptons, such as [20], can predict large LFVHD. So when calculating the LFVHD in SUSY versions, the non-SUSY contributions must be included. In contrast, the 3-3-1 models with light leptons [21] give suppressed signals of LFVHD, and the SUSY-contributions in [44] are dominant.

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Appendix A. Master integrals for one-loop integral calculation

A.1. Master integrals

The calculation in this section relates with one-loop diagrams in the Fig. 1. We introduce the notations $D_0 = k^2 - M_0^2 + i\delta$, $D_1 = (k - p_1)^2 - M_1^2 + i\delta$ and $D_2 = (k + p_2)^2 - M_2^2 + i\delta$, where δ is infinitesimally a positive real quantity. The scalar integrals are defined as

$$
\begin{aligned}
A_0(M_i) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k}{D_i}, \qquad B_0^{(1)} \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k}{D_0 D_1},\\
B_0^{(2)} &\equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k}{D_0 D_2}, \qquad B_0^{(12)} \equiv \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k}{D_1 D_2},\\
C_0 &\equiv C_0(M_0, M_1, M_2) = \frac{1}{i\pi^2}\int \frac{d^4 k}{D_0 D_1 D_2},
\end{aligned} \tag{A.1}
$$

where $i = 1, 2$. In addition, $D = 4 - 2\epsilon \le 4$ is the dimension of the integral. The notations M_0, M_1, M_2 are masses of virtual particles in the loops. The momenta satisfy conditions: $p_1^2 = m_1^2,\ p_2^2 = m_2^2$, and $(p_1 + p_2)^2 = m_{h^0}^2$. The tensor integrals are

$$
\begin{aligned}
A^\mu(p_i; M_i) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k \times k^\mu}{D_i} = A_0(M_i)p_i^\mu,\\
B^\mu(p_i; M_0, M_i) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k \times k^\mu}{D_0 D_i} \equiv B_1^{(i)} p_i^\mu,\\
B^\mu(p_1, p_2; M_1, M_i) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int \frac{d^D k \times k^\mu}{D_1 D_2} \equiv B_1^{(12)} p_1^\mu + B_2^{(12)} p_2^\mu,
\end{aligned}
$$

$$C^{\mu} = C^{\mu}(M_0, M_1, M_2) = \frac{1}{i\pi^2}\int \frac{d^4k \times k^{\mu}}{D_0 D_1 D_2} \equiv C_1 p_1^{\mu} + C_2 p_2^{\mu}, \tag{A.2}$$

where A_0, $B^{(i)}_{0,1}$, $B^{(12)}_i$ and $C_{0,1,2}$ are PV-functions. It is well-known that C_i is finite while the remains are divergent. We define

$$\Delta_\epsilon \equiv \frac{1}{\epsilon} + \ln 4\pi - \gamma_E + \ln \frac{\mu^2}{m_h^2}, \tag{A.3}$$

where γ_E is the Euler constant and m_h is the mass of the neutral Higgs. The divergent parts of the above scalar factors can be determined as

$$\begin{aligned} &\mathrm{Div}[A_0(M_i)] = M_i^2 \Delta_\epsilon, \quad \mathrm{Div}[B_0^{(i)}] = \mathrm{Div}[B_0^{(12)}] = \Delta_\epsilon, \\ &\mathrm{Div}[B_1^{(1)}] = \mathrm{Div}[B_1^{(12)}] = \frac{1}{2}\Delta_\epsilon, \quad \mathrm{Div}[B_1^{(2)}] = \mathrm{Div}[B_2^{(12)}] = -\frac{1}{2}\Delta_\epsilon. \end{aligned} \tag{A.4}$$

We remind that the finite parts of the PV-functions such as B-functions depend on the scale of μ parameter with the same coefficient of the divergent parts.

The analytic formulas of the above PV-functions are:

$$A_0(M) = M^2\left(\Delta_\epsilon + \ln\frac{m_h^2 - i\delta}{M^2 - i\delta} + 1\right) \equiv M^2\Delta_\epsilon + a_0(M), \tag{A.5}$$

$$B^{(i)}_{0,1} = \mathrm{Div}[B^{(i)}_{0,1}] + b^{(i)}_{0,1}, \quad B^{(12)}_{0,1,2} = \mathrm{Div}[B^{(12)}_{0,1,2}] + b^{(12)}_{0,1,2}, \tag{A.6}$$

where

$$\begin{aligned} b_0^{(i)} &= \ln(m_h^2 - i\delta) - \int_0^1 dx \ln\left[x^2 p_i^2 - x(p_i^2 + M_0^2 - M_i^2) + M_0^2 - i\delta\right], \\ b_0^{(12)} &= \ln(m_h^2 - i\delta) - \int_0^1 dx \ln\left[m_h^2 x^2 - x(m_h^2 + M_1^2 - M_2^2) + M_1^2 - i\delta\right]. \end{aligned} \tag{A.7}$$

The $b_0^{(1)}$ can be found in a very simple form in the limit $p_i^2 \to 0$. The $b_0^{(12)}$ is determined by

$$b_0^{(12)} = -\sum_{k=1}^{2}\int_0^1 dx \ln(x - x_k), \tag{A.8}$$

where x_k $(k = 1, 2)$ are solutions of the equation

$$x^2 - \left(\frac{m_h^2 - M_1^2 + M_2^2}{m_h^2}\right)x + \frac{M_2^2 - i\delta}{m_h^2} = 0. \tag{A.9}$$

The final expression of $b_0^{(12)}$ is

$$b_0^{(12)} = \ln\frac{m_h^2 - i\delta}{M_1^2 - i\delta} + 2 + \sum_{k=1}^{2} x_k \ln\left(1 - \frac{1}{x_k}\right). \tag{A.10}$$

The B_1^i, $B_i^{(12)}$ are calculated through the B_0 and A_0 functions, namely

$$B_1^{(i)} = \frac{(-1)^{i-1}}{2m_i^2}\left[A_0(M_i) - A_0(M_0) + B_0^{(i)}(M_0^2 - M_i^2 + m_i^2)\right],$$
$$B_i^{(12)} = \frac{1}{2m_h^2}\left[A_0(M_1) - A_0(M_2) + B_0^{(12)}\left(M_2^2 - M_1^2 + (-1)^{i-1}m_h^2\right)\right]. \tag{A.11}$$

The C_i functions can be found through the equation

$$\begin{pmatrix} 2m_1^2 & m_h^2 - m_1^2 - m_2^2 \\ m_h^2 - m_1^2 - m_2^2 & 2m_2^2 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$
$$= \begin{pmatrix} B_0^{(12)} - B_0^{(2)} + (M_0^2 - M_1^2 + m_1^2)C_0 \\ -\left[B_0^{(12)} - B_0^{(1)} + (M_0^2 - M_2^2 + m_2^2)C_0\right] \end{pmatrix}. \tag{A.12}$$

The C_0 function was generally calculated in [45], a more explicit explanation was given in [46]. In the limit $p_1^2, p_2^2 \to 0$, we get the following expression

$$\begin{aligned} C_0 &= -\int_0^1 dx \int_0^{1-x} \frac{dy}{(1-x-y)M_0^2 + xM_1^2 + yM_2^2 - xym_h^2 - i\delta} \\ &= \frac{1}{m_h^2}\int_0^1 \frac{dx}{x-x_0} \\ &\times \left[\ln\frac{m_h^2 - i\delta'}{M_1^2 - M_0^2 - i\delta'} + \ln(x-x_1) + \ln(x-x_2) - \ln(x-x_3)\right] \\ &= \frac{1}{m_h^2}\ln\frac{m_h^2 - i\delta'}{M_1^2 - M_0^2 - i\delta'} \times \ln\left(1 - \frac{1}{x_0}\right) \\ &+ \frac{1}{m_h^2}\int_0^1 \frac{dx}{x-x_0}\left[\ln(x-x_1) + \ln(x-x_2) - \ln(x-x_3)\right], \end{aligned} \tag{A.13}$$

where both δ and δ' are positive and extremely small, x_0 and x_3 are defined as

$$x_0 = \frac{M_2^2 - M_0^2}{m_h^2}, \qquad x_3 = \frac{-M_0^2 + i\delta}{M_1^2 - M_0^2}, \tag{A.14}$$

and x_1, x_2 are solutions of the equation (A.9). The limit of $p_1^2, p_2^2 = 0$ will be used in our work, even when the loops contain active neutrinos with masses extremely smaller than these quantities, because of the appearance of heavy virtual particles. The explanation is as follows. The denominator in the first line of (A.13) has the general form of $D = (1-x-y)M_0^2 + xM_1^2 + yM_2^2 - xym_h^2 - i\delta - (1-x-y)\left[xm_1^2 + ym_2^2\right]$. Our calculation relates to the two following cases:

- Only M_0 is the mass of the active neutrino, $M_0 \ll M_1, M_2$. We have $D = (1-x-y)M_0^2 + xM_1^2\left[1 - (1-x-y)m_1^2/M_1^2\right] + yM_2^2\left[1 - (1-x-y)m_2^2/M_2^2\right] - xym_h^2 - i\delta \simeq (1-x-y)M_0^2 + xM_1^2 + yM_2^2 - xym_h^2 - i\delta$.

- $M_1 = M_2$ is the mass of the neutrino: $M_1 = M_2 \ll M_0$. Then we have $D = (1-x-y)M_0^2\left[1-(xm_1^2+ym_2^2)/M_0^2\right] + xM_1^2 + yM_2^2 - xym_h^2 - i\delta \simeq (1-x-y)M_0^2 + xM_1^2 + yM_2^2 - xym_h^2 - i\delta$.

We use the following result given in [45]

$$R(x_0, x_i) \equiv \int_0^1 \frac{dx}{x-x_0}\left[\ln(x-x_i) - \ln(x_0-x_i)\right]$$
$$= Li_2(\frac{x_0}{x_0-x_i}) - Li_2(\frac{x_0-1}{x_0-x_i}), \tag{A.15}$$

where $i = 1, 2, 3$ and $Li_2(z)$ is the di-logarithm defined by

$$Li_2(z) \equiv \int_0^1 -\frac{dt}{t}\ln(1-tz).$$

We also use the real values of x_0 to give the result $\eta(-x_i, \frac{1}{x_0-x_i})\ln\frac{x_0}{x_0-x_i} = \eta(1-x_i, \frac{1}{x_0-x_i}) \times \ln\frac{x_0-1}{x_0-x_i} = 0$ for any complex x_i. Now we introduce the function

$$R_0(x_0, x_i) \equiv Li_2(\frac{x_0}{x_0-x_i}) - Li_2(\frac{x_0-1}{x_0-x_i}), \tag{A.16}$$

leading to

$$\int_0^1 \frac{dx\,\ln(x-x_i)}{x-x_0} = R_0(x_0, x_i) + \ln\left(1-\frac{1}{x_0}\right)\ln(x_0-x_i). \tag{A.17}$$

Using the following equalities

$$\ln(AB - i\delta) = \ln(A - i\delta') + \ln(B - i\delta/A)$$

with any real A, B, δ, δ' positive real and extremely small; and

$$x_1x_2 = \frac{m_h^2 - M_1^2 + M_2^2}{m_h^2}, \qquad x_1x_2 = \frac{M_2^2 - i\delta}{m_h^2},$$

we can prove that

$$\ln\frac{m_h^2 - i\delta'}{M_1^2 - M_0^2 - i\delta'} + \ln(x_0-x_1) + \ln(x_0-x_2) - \ln(x_0-x_3) = 0.$$

This results the very simple expression of C_0 function

$$C_0 = \frac{1}{m_h^2}\left[R_0(x_0, x_1) + R_0(x_0, x_2) - R_0(x_0, x_3)\right], \tag{A.18}$$

where $x_{1,2}$ are solutions of the equation (A.9), and $x_{0,3}$ are given in (A.14). This result is consistent with that discussed on [31].

For simplicity in calculation we will also use other approximations of PV-functions where $p_1^2, p_2^2 \to 0$, namely

$$a_0(M) = M^2\left(1+\ln\frac{m_h^2-i\delta}{M^2-i\delta}\right), \qquad b_0^{(i)} = 1-\ln\frac{M_i^2}{m_h^2}+\frac{M_0^2}{M_0^2-M_i^2}\ln\frac{M_i^2}{M_0^2},$$

$$b_1^{(1)} = -\frac{1}{2}\ln\frac{M_1^2}{m_h^2}-\frac{M_0^4}{2(M_0^2-M_1^2)^2}\ln\frac{M_0^2}{M_1^2}+\frac{(M_0^2-M_1^2)(3M_0^2-M_1^2)}{4(M_0^2-M_1^2)^2},$$

$$b_1^{(2)} = \frac{1}{2}\ln\frac{M_2^2}{m_h^2}+\frac{M_0^4}{2(M_0^2-M_2^2)^2}\ln\frac{M_0^2}{M_2^2}-\frac{(M_0^2-M_2^2)(3M_0^2-M_2^2)}{4(M_0^2-M_2^2)^2},$$

$$b_0^{(12)} = \ln\frac{m_h^2-i\delta}{M_1^2-i\delta}+2+\sum_{k=1}^{2}x_k\ln\left(1-\frac{1}{x_k}\right),$$

where x_k is the two solutions of the equation (A.9),

$$b_i^{(12)} = \frac{1}{2m_h^2}\left[M_1^2\left(1+\ln\frac{m_h^2}{M_1^2}\right)-M_2^2\left(1+\ln\frac{m_h^2}{M_2^2}\right)\right] + \frac{b_0^{(12)}}{2m_h^2}\left[M_2^2-M_1^2+(-1)^{i-1}m_h^2\right],$$

$$C_1 = \frac{1}{m_h^2}\left[b_0^{(1)}-b_0^{(12)}+(M_2^2-M_0^2)C_0\right],$$

$$C_2 = -\frac{1}{m_h^2}\left[b_0^{(2)}-b_0^{(12)}+(M_1^2-M_0^2)C_0\right].$$

Appendix B. Calculations the one loop contributions

In the first part of this section we will calculate in details the contributions of particular contributions of diagrams shown in the Fig. 1 which involve with exotic neutral lepton N_a, $a=1,2,3$. From this we can derive the general functions expressing the contributions of particular diagrams.

B.1. Amplitudes

It is needed to remind that the amplitude will be expressed in terms of the PV-functions, so the integral will be written as

$$\int\frac{d^4k}{(2\pi)^4}\to\frac{i}{16\pi^2}\times\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^4k,$$

where μ is a parameter with dimension of mass. This step will be omitted in the below calculation, the final results are simply corrected by adding the factor $i/16\pi^2$. As an example in the calculation of contribution from the first diagram, we will point out a class of divergences that automatically vanish by the GIM mechanism. More explicitly for any terms which do not depend on the masses of virtual leptons, they will vanish because of the appearance of the factor $\sum_a V_{1a}^L V_{2a}^{L*}=0$.

The contribution from diagram 1a) is:

$$
\begin{aligned}
i\mathcal{M}^{FVV}_{(a)} =& \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1 \frac{ig}{\sqrt{2}} V_{1a}\gamma^{\mu} P_L \frac{1(\not{k}+m_a)}{D_0} \frac{ig}{\sqrt{2}} V^*_{2a}\gamma^{\nu} P_L v_2 \\
&\times \left[\frac{igm_V}{\sqrt{2}}\left(-c_\alpha s_\theta + \sqrt{2}s_\alpha c_\theta\right)\right]\frac{-i}{D_1} \\
&\times \left[g_{\mu\alpha} - \frac{(k-p_1)_\mu (k-p_1)_\alpha}{m_V^2}\right]\frac{-i}{D_2}\left[g_{\nu\beta} - \frac{(k+p_2)_\nu (k+p_2)_\beta}{m_V^2}\right] \\
=& \sum_a V_{1a}V^*_{2a}(-1)\frac{g^3 m_V}{2\sqrt{2}}\left(-c_\alpha s_\theta + \sqrt{2}s_\alpha c_\theta\right) \times \int \frac{d^4k}{(2\pi)^4}\frac{\bar{u}_1\gamma^{\mu}\not{k}\gamma^{\nu}P_L v_2}{D_0 D_1 D_2} \\
&\times g^{\alpha\beta}\left[g_{\mu\alpha} - \frac{(k-p_1)_\mu (k-p_1)_\alpha}{m_V^2}\right]\left[g_{\beta\nu} - \frac{(k+p_2)_\nu (k+p_1)_\beta}{m_V^2}\right] \\
\equiv& \sum_a V_{1a}V^*_{2a}(-1)\frac{g^3 m_V}{2\sqrt{2}}\left(-c_\alpha s_\theta + \sqrt{2}s_\alpha c_\theta\right)[P_1 + P_2 + P_3],
\end{aligned}
\tag{B.1}
$$

where

$$
\begin{aligned}
P_1 =& \int \frac{d^4k}{(2\pi)^4}\frac{\bar{u}_1\gamma^{\mu}\not{k}\gamma^{\nu}P_L v_2}{D_0 D_1 D_2} g_{\mu\nu} = \int \frac{d^4k}{(2\pi)^4}\frac{(2-d)\bar{u}_1\not{k}P_L v_2}{D_0 D_1 D_2} \\
=& \bar{u}_1 P_L v_2 \times m_1(-2C_1) + \bar{u}_1 P_L v_2 \times m_2(2C_2).
\end{aligned}
\tag{B.2}
$$

We can see that P_1 does not contain any divergent terms. The formula of P_2 is

$$
\begin{aligned}
P_2 =& \frac{-1}{m_V^2}\int \frac{d^4k}{(2\pi)^4}\frac{\bar{u}_1\gamma^{\mu}\not{k}\gamma^{\nu}P_L v_2}{D_0 D_1 D_2}\left[(k+p_2)_\mu (k+p_2)_\nu + (k-p_1)_\mu (k-p_1)_\nu\right] \\
=& \frac{-1}{m_V^2}\int \frac{d^4k}{(2\pi)^4}\left[\frac{\bar{u}_1(D_0+m_a^2)(\not{k}+2\not{p}_2)P_L v_2 + \bar{u}_1\not{p}_2\not{k}\not{p}_2 P_L v_2}{D_0 D_1 D_2}\right. \\
&\left. + \frac{\bar{u}_1(D_0+m_a^2)(\not{k}-2\not{p}_1)P_L v_2 + \bar{u}_1\not{p}_1\not{k}\not{p}_1 P_L v_2}{D_0 D_1 D_2}\right] \\
=& \frac{-1}{m_V^2}\left\{\bar{u}_1 P_L v_2 \times m_1\left[2B_1^{(12)}(m_V) - 2B_0^{(12)}(m_V)\right.\right. \\
&\left. - 2m_a^2 C_0 + (2m_a^2 + m_1^2 - m_2^2)C_1 + (m_{H_0}^2 - m_1^2 - m_2^2)C_2\right] \\
&+ \bar{u}_1 P_R v_2 \times m_2\left[-2B_2^{(12)}(m_V) - 2B_0^{(12)}(m_V)\right. \\
&\left.\left. - 2m_a^2 C_0 - (2m_a^2 - m_1^2 + m_2^2)C_2 - (m_{H_0}^2 - m_1^2 - m_2^2)C_1\right]\right\}.
\end{aligned}
\tag{B.3}
$$

We can see that the terms like $B_1^{(12)}(m_V)$, $B_1^{(12)}(m_V)$ and $B_0^{(12)}(m_V)$ do contain divergences but they do not depend on m_a in the loop. Hence these terms will exactly cancel by the GIM mechanism. All of the other are finite.

The contribution from P_3 is

$$P_3 = \frac{1}{m_V^4}\int \frac{d^4k}{(2\pi)^4} \times \frac{\bar{u}_1\gamma^{\mu}\not{k}\gamma^{\nu}P_L v_2}{D_0D_1D_2}\left[(k-p_1).(k+p_2)(k-p_1)_{\mu}(k+p_2)_{\nu}\right]$$
$$= \frac{1}{2m_V^4}\int \frac{d^4k}{(2\pi)^4} \times \left[\frac{\bar{u}_1[D_1+D_2+2m_V^2-m_{H_0}^2](D_0+m_a^2)(\not{k}+\not{p}_2-\not{p}_1)P_L v_2}{D_0D_1D_2}\right.$$
$$\left. + m_1m_2\frac{\bar{u}_1[D_1+D_2+2m_V^2-m_{H_0}^2]\not{k}P_L v_2}{D_0D_1D_2}\right]$$
$$= \frac{1}{2m_V^4}\left\{ \bar{u}_1P_Lv_2 \times m_1\left[-A_0(m_V)+(2m_V^2-m_{H_0}^2)\left(B_1^{(12)}(m_V)-B_0^{(12)}(m_V)\right)\right.\right.$$
$$- m_2^2B_1^{(2)} + \mathbf{m_a^2}\left(\mathbf{B_1^{(1)}} - \mathbf{B_0^{(1)}} - \mathbf{B_0^{(2)}}\right)$$
$$\left. + (2m_V^2-m_{H_0}^2)\left(m_a^2(C_1-C_0)-m_2^2C_2\right)\right]$$
$$+ \bar{u}_1P_Rv_2 \times m_2\left[-A_0(m_V)+(2m_V^2-m_{H_0}^2)\left(-B_2^{(12)}(m_V)-B_0^{(12)}(m_V)\right)\right.$$
$$+ m_1^2B_1^{(1)} + \mathbf{m_a^2}\left(-\mathbf{B_0^{(1)}} - \mathbf{B_0^{(2)}} - \mathbf{B_1^{(2)}}\right)$$
$$\left.\left. + (2m_V^2-m_{H_0}^2)\left(m_1^2C_1-m_a^2(C_0+C_2)\right)\right]\right\}. \tag{B.4}$$

Again all terms in the first and third lines do not contribute to the amplitude. But the four terms $m_2^2B_1^{(2)}$, $m_1^2B_1^{(1)}$, $m_a^2\left(B_1^{(1)}-B_0^{(1)}-B_0^{(2)}\right)$ and $m_a^2\left(-B_0^{(1)}-B_0^{(2)}-B_1^{(2)}\right)$ do contain divergences. The first two terms have divergent parts having the corresponding forms of $(-m_2^2\Delta_\epsilon)$ and $m_1^2\Delta_\epsilon$, which do not depend on the masses m_a of the virtual leptons. Hence they also vanish by the GIM mechanism. The finite parts of these terms still contribute to the amplitude. The remain two terms include the most dangerous divergent parts. They have factors m_a^2 which can not cancel by the GIM mechanism. We remark them by the bold and will prove later that they finally vanish after summing all diagrams. From now on we can exclude all terms that do not depend on the masses of virtual leptons.

Based on definition $\mathcal{M} = -\left(E_L^{FVV}\overline{u_1}P_Lv_2 + E_R^{FVV}\overline{u_1}P_Rv_2\right)$, the expression of the total contribution from the diagram 1a) is simply

$$\mathcal{M}_{(a)}^{FVV} = \frac{-g^3}{32\pi^2\sqrt{2}}\left(-c_\alpha s_\theta + \sqrt{2}s_\alpha c_\theta\right)\sum_a V_{1a}V_{2a}^*\left[(\overline{u_1}P_Lv_2)\,E_L^{FVV} + (\overline{u_1}P_Rv_2)\,E_R^{FVV}\right], \tag{B.5}$$

where $E_{L,R}^{FVV}$ is defined in (4) and (5). Here we have added a factor of $\frac{i}{16\pi^2}$. All terms being independent on m_a will cancel by the factor $\sum_a V_{1a}V_{2a}^*$. If we assume all other divergences cancel among themselves after summing all of the diagrams, the analytic formulas of E_L^{FVV} and E_R^{FVV} can be written in terms of the finite parts of PV-functions, i.e. $b_0^{(i)}$, $b_0^{(12)}$, b_1^i, $b_i^{(12)}$ and $C_{0,1,2}$. The following calculation for the remain diagrams will be done the same as what we have done above. We trace the divergence of each diagram in the bold text.

The contribution from diagram 1b) is:

$$
\begin{aligned}
i\mathcal{M}^{FVH}_{(b)} &= \sum_a \int \frac{d^4k}{(2\pi)^4} \\
&\times \bar{u}_1 \frac{ig}{\sqrt{2}} V_{1a}\gamma^{\mu} P_L \frac{i(\not{k}+m_a)}{D_0}(-i\sqrt{2}V^*_{2a})\left(\frac{m_2}{v_1}a_1P_R + \frac{m_a}{v_3}a_3P_L\right)v_2 \\
&\quad \times \frac{ig}{2\sqrt{2}}(-k-2p_2-p_1)^{\alpha}\frac{i}{D_2}\frac{-i}{D_1}\left[g_{\mu\alpha} - \frac{(k-p_1)_{\mu}(k-p_1)_{\alpha}}{m_V^2}\right] \\
&= \sum_a V_{1a}V^*_{2a}\frac{g^2}{2\sqrt{2}}(c_{\alpha}c_{\theta}+\sqrt{2}s_{\alpha}s_{\theta}) \\
&\quad \times \int \frac{d^4k}{(2\pi)^4} \times \frac{\frac{m_2}{v_1}a_1\bar{u}_1\gamma^{\mu}\not{k}P_Rv_2 + \frac{m_a^2}{v_2}a_2\bar{u}_1\gamma^{\mu}P_Lv_2}{D_0D_1D_2} \\
&\quad \times \left[(k+2p_2+p_1)_{\mu} - \frac{(k+2p_2+p_1).(k-p_1)(k-p_1)_{\mu}}{m_V^2}\right] \\
&= \sum_a V_{1a}V^*_{2a}\left[\frac{g^2}{2\sqrt{2}}(c_{\alpha}c_{\theta}+\sqrt{2}s_{\alpha}s_{\theta})\right] \\
&\quad \times \Bigg\{ \bar{u}_1P_Lv_2 \times \Bigg[\boxed{\mathbf{\frac{-m_1}{m_V^2}\frac{m_a^2}{v_3}a_3\left(B_1^{(1)}-B_0^{(1)}\right)}} \\
&\quad + \frac{m_a^2}{v_3}a_3 \times m_1\left(C_0+C_1+\frac{(m^2_{H_A}-m^2_{H_0})}{m_V^2}(C_0-C_1)\right) \\
&\quad + \frac{m_2}{v_1}a_1 \times m_1m_2\left(2C_1-C_2-\frac{m^2_{H_A}-m^2_{H_0}}{m_V^2}C_2\right)\Bigg] \\
&\quad + \bar{u}_1P_Rv_2 \times \Bigg[\frac{-1}{m_V^2}\frac{m_2}{v_1}a_1\left(A_0(m_V)+(m^2_{H_A}-m^2_{H_0})B_0^{(12)}\right) \\
&\quad + \frac{m_2}{v_1}a_1B_0^{(12)} + \frac{m_1^2}{m_V^2}\frac{m_2}{v_1}a_1B_1^{(1)} - \boxed{\mathbf{\frac{m_a^2}{m_V^2}\frac{m_2}{v_1}a_1B_0^{(1)}(m_a,m_V)}} \\
&\quad + \frac{m_2}{v_1}a_1\Bigg(m_a^2C_0 - m_1^2C_1 + 2m_2^2C_2 + 2(m^2_{H_0}-m_2^2)C_1 \\
&\quad - \frac{(m^2_{H_A}-m^2_{H_0})}{m_V^2}\left(m_a^2C_0-m_1^2C_1\right)\Bigg) \\
&\quad + \frac{m_a^2}{v_3}a_3 \times m_2\left(-2C_0-C_2+\frac{(m^2_{H_A}-m^2_{H_0})}{m_V^2}C_2\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{B.6}
$$

The contribution to the total amplitude is

$$\mathcal{M}_{(b)}^{FVH} = \frac{g^2}{32\pi^2\sqrt{2}} \left(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta\right) \sum_a V_{1a} V_{2a}^* \left[(\overline{u_1} P_L v_2)\, E_L^{FVH} + (\overline{u_1} P_R v_2)\, E_R^{FVH} \right]. \tag{B.7}$$

The contribution from diagram 1c) is:

$$\begin{aligned}
i\mathcal{M}_{(c)}^{FHV} &= \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1(-i\sqrt{2}V_{1a}) \left(\frac{m_1}{v_1} a_1 P_L + \frac{m_a}{v_3} a_3 P_R\right) \\
&\quad \times \frac{i(\not{k} + m_a)}{D_0} \frac{ig}{\sqrt{2}} V_{2a}^* \gamma^\mu P_L v_2 \times \frac{ig}{2\sqrt{2}} (c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta)(-k + p_2 + 2p_1)^\alpha \\
&\quad \times \frac{i}{D_1} \frac{-i}{D_2} \times \left[g_{\mu\alpha} - \frac{(k+p_2)_\mu (k+p_2)_\alpha}{m_V^2} \right] \\
&= \sum_a V_{1a} V_{2a}^* \frac{g^2}{2\sqrt{2}} (c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta) \int \frac{d^4k}{(2\pi)^4} \\
&\quad \times \left[\frac{m_1}{v_1} a_1 \frac{\bar{u}_1 \gamma^\mu \not{k} P_L v_2}{D_0 D_1 D_2} + \frac{m_a^2}{v_3} a_3 \frac{\bar{u}_1 \gamma^\mu P_L v_2}{D_0 D_1 D_2} \right] \\
&\quad \times \left[(k - p_2 - 2p_1)_\mu - \frac{(k - p_2 - 2p_1).(k+p_2)(k+p_2)_\mu}{m_V^2} \right] \\
&= \sum_a V_{1a} V_{2a}^* \left[\frac{g^2}{2\sqrt{2}} (c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta) \right] V_{1a} V_{2a}^* \\
&\quad \times \left\{ \bar{u}_1 P_L v_2 \times \left[\frac{-1}{m_V^2} \frac{m_1}{v_1} a_1 \left(A_0(m_V) + (m_{H_A}^2 - m_{H_0}^2) B_0^{(12)} \right) \right.\right. \\
&\quad + \frac{m_1}{v_1} a_1 B_0^{(12)}(m_V, m_{H_A}) - \frac{m_2^2}{m_V^2} \frac{m_1}{v_1} a_1 B_1^{(2)}(m_a, m_V) \\
&\quad - \boxed{ \mathbf{\frac{m_a^2}{m_V^2} \frac{m_1}{v_1} a_1 B_0^{(2)}(m_a, m_V)} } \\
&\quad + \frac{m_1}{v_1} a_1 \left(m_a^2 C_0 - 2m_1^2 C_1 + m_2^2 C_2 - 2(m_{H_0}^2 - m_1^2) C_2 \right. \\
&\quad \left. - \frac{(m_{H_A}^2 - m_{H_0}^2)}{m_V^2} (m_2^2 C_2 + m_a^2 C_0) \right) \\
&\quad \left. + m_1 \frac{m_a^2}{v_3} a_3 \left(-2C_0 + C_1 - \frac{(m_{H_A}^2 - m_{H_0}^2)}{m_V^2} C_1 \right) \right] \\
&\quad + \bar{u}_1 P_R v_2 \left[\boxed{ \mathbf{\frac{m_2}{m_V^2} \frac{m_a^2}{v_3} a_3 \left(B_1^{(2)} + B_0^{(2)} \right)} } \right.
\end{aligned}$$

$$+ m_1 m_2 \frac{m_1}{v_1} a_1 \left(C_1 - 2C_2 + \frac{(m_{H_A}^2 - m_{H_0}^2)}{m_V^2} C_1 \right)$$
$$+ m_2 \frac{m_a^2}{v_3} a_3 \left(C_0 - C_2 + \frac{(m_{H_A}^2 - m_{H_0}^2)}{m_V^2} (C_0 + C_2) \right) \Bigg] \Bigg] \Bigg\}. \tag{B.8}$$

The contribution to the total amplitude is

$$\mathcal{M}_{(c)}^{FHV} = \frac{g^2}{32\pi^2\sqrt{2}} \left(c_\alpha c_\theta + \sqrt{2} s_\alpha s_\theta \right) \sum_a V_{1a} V_{2a}^* \left[(\overline{u_1} P_L v_2)\, E_L^{FHV} + (\overline{u_1} P_R v_2)\, E_R^{FHV} \right]. \tag{B.9}$$

The contribution from diagram 1d) is:

$$\begin{aligned}
i\mathcal{M}_{(d)}^{FHH} &= \sum_a \int \frac{d^4k}{(2\pi)^4} \times (-i v_3 \lambda_{h^0 H_1 H_1}) \frac{i}{D_1} \frac{i}{D_2} \times \bar{u}_1 (-i\sqrt{2} V_{1a}) \\
&\times \left(\frac{m_1}{v_1} a_1 P_L + \frac{m_a}{v_3} a_3 P_R \right) \frac{i(\not{k} + m_a)}{D_0} (-i\sqrt{2} V_{2a}^*) \left(\frac{m_2}{v_1} a_1 P_R + \frac{m_a}{v_3} a_3 P_L \right) v_2 \\
&= \sum_a v_3 \lambda_{h^0 H_1 H_1} V_{1a} V_{2a}^* \int \frac{d^4k}{(2\pi)^4} \\
&\times \frac{\bar{u}_1 \left(\frac{m_1}{v_1} a_1 P_L + \frac{m_a}{v_3} a_3 P_R \right) (\not{k} + m_a) \left(\frac{m_2}{v_1} a_1 P_R + \frac{m_a}{v_3} a_3 P_L \right) v_2}{D_0 D_1 D_2} \\
&= \sum_a v_3 \lambda_{h^0 H_1 H_1} V_{1a} V_{2a}^* \int \frac{d^4k}{(2\pi)^4} \\
&\times \left[\frac{m_1 m_2}{v_1^2} a_1^2 \frac{\bar{u}_1 \not{k} P_R v_2}{D_0 D_1 D_2} + \frac{m_1 m_a^2}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 P_L v_2}{D_0 D_1 D_2} \right. \\
&\left. + \frac{m_a^2}{v_3^2} a_3^2 \frac{\bar{u}_1 \not{k} P_L v_2}{D_0 D_1 D_2} + \frac{m_2 m_a^2}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 P_R v_2}{D_0 D_1 D_2} \right] \\
&= \sum_a v_3 \lambda_{h^0 H_1 H_1} V_{1a} V_{2a}^* \\
&\times \left\{ \bar{u}_1 P_L v_2 \times m_1 \left[\frac{m_a^2}{v_1 v_3} a_1 a_3 C_0 - \frac{m_2^2}{v_1^2} a_1^2 C_2 + \frac{m_a^2}{v_3^2} a_3^2 C_1 \right] \right. \\
&\left. + \bar{u}_1 P_R v_2 \times m_2 \left[\frac{m_a^2}{v_1 v_3} a_1 a_3 C_0 + \frac{m_1^2}{v_1^2} a_1^2 C_1 - \frac{m_a^2}{v_3^2} a_3^2 C_2 \right] \right\}
\end{aligned} \tag{B.10}$$

with $\lambda_{h^0 H_1 H_1}$ shown in the Table 1. With the notations of E_L^{FHH} and E_R^{FHH} defined in (10) and (11), the contribution to the amplitude is

$$\mathcal{M}_{(d)}^{FHH} = \frac{1}{64\pi^2\sqrt{2}} \times (4\sqrt{2}\lambda_{h^0 H_1 H_1}) \sum_a V_{1a} V_{2a}^* \left[(\overline{u_1} P_L v_2)\, E_L^{FHH} + (\overline{u_1} P_R v_2)\, E_R^{FHH} \right]. \tag{B.11}$$

The contribution from diagram 1e) is:

$$
\begin{aligned}
i\mathcal{M}^{VFF}_{(e)} &= \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1 \frac{ig}{\sqrt{2}} V_{1a}\gamma^\mu P_L \frac{i(-\not{k}+\not{p}_1+m_a)}{D_1}\left(\frac{-igm_a}{2m_V}\frac{s_\alpha}{c_\theta}\right) \\
&\quad \times \frac{i(-\not{k}-\not{p}_2+m_a)}{D_2}\frac{ig}{\sqrt{2}} V^*_{2a}\gamma^\nu P_L v_2 \frac{-i}{D_0}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{m_V^2}\right] \\
&= \sum_a -\frac{g^3 m_a}{4m_V}\frac{s_\alpha}{c_\theta} V_{1a}V^*_{2a} \int \frac{d^4k}{(2\pi)^4}\left[\frac{(2-d)m_a\bar{u}_1(-2\not{k}+\not{p}_1-\not{p}_2)P_L v_2}{D_0D_1D_2}\right. \\
&\quad \left. - \frac{m_a}{m_V^2}\frac{\bar{u}_1\not{k}(-2\not{k}+\not{p}_1-\not{p}_2)\not{k}P_L v_2}{D_0D_1D_2}\right] \\
&= \sum_a \left[-\frac{g^3 m_a}{4m_V}\frac{s_\alpha}{c_\theta}\right] V_{1a}V^*_{2a}\left\{\bar{u}_1 P_L v_2 \times m_1 m_a \left[\boxed{\frac{\mathbf{1}}{\mathbf{m_V^2}}\left(\mathbf{B_0^{(12)}} + \mathbf{B_1^{(1)}}\right)}\right.\right. \\
&\quad \left. - \frac{1}{m_V^2}\left(-m_V^2 C_0 + (m_1^2+m_2^2-2m_a^2)C_1\right) + (2-d)(C_0-2C_1)\right] \\
&\quad + \bar{u}_1 P_R v_2 \times m_2 m_a \left[\boxed{\frac{\mathbf{1}}{\mathbf{m_V^2}}\left(\mathbf{B_0^{(12)}} - \mathbf{B_1^{(2)}}\right)} + (2-d)(C_0+2C_2)\right. \\
&\quad \left.\left. - \frac{1}{m_V^2}\left(-m_V^2 C_0 - (m_1^2+m_2^2-2m_a^2)C_2\right)\right]\right\}.
\end{aligned} \tag{B.12}
$$

The final result is written as

$$
\mathcal{M}^{VFF}_{(e)} = \left[-\frac{1}{64\pi^2\sqrt{2}} \times \frac{g^3 s_\alpha\sqrt{2}}{c_\theta}\right]\sum_a V_{1a}V^*_{2a}\left[(\overline{u_1}P_L v_2)\, E_L^{VFF} + (\overline{u_1}P_R v_2)\, E_R^{VFF}\right], \tag{B.13}
$$

where $E^{VFF}_{L,R}$ are defined in (12) and (13).

The contribution from diagram 1f) is

$$
\begin{aligned}
i\mathcal{M}^{HFF}_{(f)} &= \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1(-i\sqrt{2}V_{1a})\left(\frac{m_1}{v_1}a_1 P_L + \frac{m_a}{v_3}a_3 P_R\right) \\
&\quad \times \frac{i(-\not{k}+\not{p}_1+m_a)}{D_1}\left(\frac{-im_a s_\alpha}{v_3}\right)\frac{i(-\not{k}-\not{p}_2+m_a)}{D_2} \\
&\quad \times (-i\sqrt{2}V^*_{2a})\left(\frac{m_2}{v_1}a_1 P_R + \frac{m_a}{v_3}a_3 P_L\right) v_2 \times \frac{i}{D_0} \\
&= \sum_a V_{1a}V^*_{2a}\left[\frac{2m_a s_\alpha}{v_3}\right]\int \frac{d^4k}{(2\pi)^4}\left[\frac{m_1 m_a}{v_1 v_3}a_1 a_3 \frac{\bar{u}_1(\not{k}-\not{p}_1)(\not{k}+\not{p}_2)P_L v_2}{D_0D_1D_2}\right. \\
&\quad + \frac{m_2 m_a}{v_1 v_3}a_1 a_3 \frac{\bar{u}_1(\not{k}-\not{p}_1)(\not{k}+\not{p}_2)P_R v_2}{D_0D_1D_2}
\end{aligned}
$$

$$
\begin{aligned}
&+ m_a \frac{m_1 m_2}{v_1^2} a_1^2 \frac{\bar{u}_1(-2\not{k} - \not{p}_2 + \not{p}_1) P_R v_2}{D_0 D_1 D_2} \\
&+ \frac{m_a^3}{v_3^2} a_3^2 \frac{\bar{u}_1(-2\not{k} - \not{p}_2 + \not{p}_1) P_L v_2}{D_0 D_1 D_2} \\
&+ \frac{m_1 m_a^3}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 P_L v_2}{D_0 D_1 D_2} + \frac{m_2 m_a^3}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 P_R v_2}{D_0 D_1 D_2} \Bigg] \\
= &\sum_a V_{1a} V_{2a}^* \left[\frac{2 m_a s_\alpha}{v_3}\right] \Bigg\{ \bar{u}_1 P_L v_2 \\
&\times m_1 m_a \left[\boxed{\mathbf{\frac{a_1 a_3}{v_1 v_3} B_0^{(12)}}} + \frac{m_2^2}{v_1^2} a_1^2 (2C_2 + C_0) + \frac{m_a^2}{v_3^2} a_3^2 (C_0 - 2C_1) \right. \\
&\left. + \frac{a_1 a_3}{v_1 v_3} \left(2 m_2^2 C_2 - (m_1^2 + m_2^2) C_1 + (m_a^2 + m_{H_A}^2 + m_2^2) C_0\right) \right] \\
&+ \bar{u}_1 P_R v_2 m_2 m_a \left[\boxed{\mathbf{\frac{a_1 a_3}{v_1 v_3} B_0^{(12)}}} + \frac{m_1^2}{v_1^2} a_1^2 (C_0 - 2C_1) + \frac{m_a^2}{v_3^2} a_3^2 (C_0 + 2C_2) \right. \\
&\left. + \frac{a_1 a_3}{v_1 v_3} \left(-2 m_1^2 C_1 + (m_1^2 + m_2^2) C_2 + (m_a^2 + m_{H_A}^2 + m_1^2) C_0 \right) \right] \Bigg\}
\end{aligned}
\tag{B.14}
$$

The final result is written as

$$
i\mathcal{M}_{(f)}^{HFF} = \frac{1}{64\pi^2\sqrt{2}} \times (8 s_\alpha \sqrt{2}) \sum_a V_{1a} V_{2a}^* \left[(\overline{u_1} P_L v_2)\, E_L^{HFF} + (\overline{u_1} P_R v_2)\, E_R^{HFF} \right], \tag{B.15}
$$

where $E_{L,R}^{HFF}$ are defined in (14) and (15).

The contribution from diagram 1g) is:

$$
\begin{aligned}
i\mathcal{M}_{(g)}^{(FV)} = &\sum_a \int \frac{d^4 k}{(2\pi)^4} \times \bar{u}_1 \left(\frac{ig}{\sqrt{2}} V_{1a} \gamma^\mu P_L\right) \frac{i(\not{k} + m_a)}{D_0} \left(\frac{ig}{\sqrt{2}} V_{2a}^* \gamma^\nu P_L\right) \\
&\times \frac{i(\not{p}_1 + m_2)}{p_1^2 - m_2^2} \left(\frac{igm_2}{2\sqrt{2} m_V} \frac{c_\alpha}{s_\theta}\right) v_2 \frac{-i}{D_1} \left[g_{\mu\nu} - \frac{(k - p_1)_\mu (k - p_1)_\nu}{m_V^2} \right] \\
= &\sum_a V_{1a} V_{2a}^* \frac{g^3}{4\sqrt{2} m_V} \frac{m_2}{(m_1^2 - m_2^2)} \frac{c_\alpha}{s_\theta} \\
&\times \int \frac{d^4 k}{(2\pi)^4} \left[\frac{(2-d)\bar{u}_1 \not{k}\not{p}_1 P_R v_2 + (2-d) m_2 \bar{u}_1 \not{k} P_L v_2}{D_0 D_1} \right. \\
&\left. - \frac{1}{m_V^2} \frac{\bar{u}_1 (\not{k} - \not{p}_1)\not{k}(\not{k} - \not{p}_1)\not{p}_1 P_R v_2}{D_0 D_1} - \frac{m_2}{m_V^2} \frac{\bar{u}_1 (\not{k} - \not{p}_1)\not{k}(\not{k} - \not{p}_1) P_L v_2}{D_0 D_1} \right] \\
= &\sum_a V_{1a} V_{2a}^* \left[\frac{g^3}{4\sqrt{2} m_V} \frac{m_2}{(m_1^2 - m_2^2)} \frac{c_\alpha}{s_\theta} \right]
\end{aligned}
$$

$$\times \left\{ \bar{u}_1 P_L v_2 \times m_1 m_2 \left[\frac{1}{m_V^2} A_0(m_V) - \frac{m_1^2}{m_V^2} B_1^{(1)} \right.\right.$$

$$\left. + (2-d) B_1^{(1)} - \boxed{ \frac{\mathbf{1}}{\mathbf{m_V^2}} \left(-\mathbf{2m_a^2 B_0^{(1)}} + \mathbf{m_a^2 B_1^{(1)}} \right) } \right]$$

$$+ \bar{u}_1 P_R v_2 \times m_1^2 \left[\frac{1}{m_V^2} A_0(m_V) - \frac{m_1^2}{m_V^2} B_1^{(1)} + (2-d) B_1^{(1)} \right.$$

$$\left.\left. - \boxed{ \frac{\mathbf{1}}{\mathbf{m_V^2}} \left(-\mathbf{2m_a^2 B_0^{(1)}} + \mathbf{m_a^2 B_1^{(1)}} \right) } \right] \right\} \tag{B.16}$$

The contribution from diagram 1h) is:

$$i\mathcal{M}_{(h)}^{VF} = \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1 \left(\frac{igm_1}{2\sqrt{2}m_V} \frac{c_\alpha}{s_\theta} \right) \frac{i(-\not{p}_2 + m_1)}{p_2^2 - m_1^2} \left(\frac{ig}{\sqrt{2}} V_{1a} \gamma^\mu P_L \right)$$

$$\times \frac{i(\not{k} + m_a)}{D_0} \left(\frac{ig}{\sqrt{2}} V_{2a}^* \gamma^\nu P_L \right) v_2 \times \frac{-i}{D_2} \left[g_{\mu\nu} - \frac{(k+p_2)_\mu (k+p_2)_\nu}{m_V^2} \right]$$

$$= \sum_a \frac{g^3}{4\sqrt{2}m_V} \frac{m_1}{(m_2^2 - m_1^2)} \frac{c_\alpha}{s_\theta} V_{1a} V_{2a}^*$$

$$\times \int \frac{d^4k}{(2\pi)^4} \left[\frac{-(2-d)\bar{u}_1 \not{p}_2 \not{k} P_L v_2 + (2-d) m_1 \bar{u}_1 \not{k} P_L v_2}{D_0 D_2} \right.$$

$$\left. + \frac{1}{m_V^2} \frac{\bar{u}_1 \not{p}_2 (\not{k} + \not{p}_2) \not{k} (\not{k} + \not{p}_2) P_L v_2}{D_0 D_2} - \frac{m_1}{m_V^2} \frac{\bar{u}_1 (\not{k} + \not{p}_2) \not{k} (\not{k} + \not{p}_2) P_L v_2}{D_0 D_2} \right]$$

$$= \sum_a \frac{g^3}{4\sqrt{2}m_V} \frac{m_1}{(m_2^2 - m_1^2)} \frac{c_\alpha}{s_\theta} V_{1a} V_{2a}^*$$

$$\times \int \frac{d^4k}{(2\pi)^4} \left[(2-d) \bar{u}_1 \left(\frac{-\not{p}_2 \not{k}}{D_0 D_2} + \frac{m_1 \not{k}}{D_0 D_2} \right) P_L v_2 \right.$$

$$+ \frac{1}{m_V^2} \bar{u}_1 \left(\frac{k^2 \not{p}_2 \not{k} + 2k^2 p_2^2 + m_2^2 \not{k} \not{p}_2}{D_0 D_2} \right) P_L v_2$$

$$\left. - \frac{m_1}{m_V^2} \bar{u}_1 \left(\frac{k^2 \not{k} + 2k^2 \not{p}_2 + \not{p}_2 \not{k} \not{p}_2}{D_0 D_2} \right) P_L v_2 \right]$$

$$= \sum_a \left[\frac{g^3}{4\sqrt{2}m_V} \frac{m_1}{(m_2^2 - m_1^2)} \frac{c_\alpha}{s_\theta} \right] V_{1a} V_{2a}^*$$

$$\times \left\{ \bar{u}_1 P_L v_2 \times m_2^2 \left[\frac{1}{m_V^2} A_0(m_V) + \frac{m_2^2}{m_V^2} B_1^{(2)} - (2-d) B_1^{(2)} \right.\right.$$

$$- \boxed{\mathbf{\frac{1}{m_V^2}\left(-2m_a^2B_0^{(2)} - m_a^2B_1^{(2)}\right)}}\Bigg]$$

$$+ \bar{u}_1 P_R v_2 \times m_1 m_2 \left[\frac{1}{m_V^2} A_0(m_V) + \frac{m_2^2}{m_V^2} B_1^{(2)} - (2-d) B_1^{(2)} \right.$$

$$\left. - \boxed{\mathbf{\frac{1}{m_V^2}\left(-2m_a^2B_0^{(2)} - m_a^2)B_1^{(2)}\right)}} \right]\Bigg\} \tag{B.17}$$

The total amplitude from the two diagrams 1g) and 1h) is:

$$i\mathcal{M}^{FV}_{(g+h)} = \sum_a \left[\frac{g^3}{4\sqrt{2}m_V}\frac{c_\alpha}{s_\theta}\right] V_{1a}V^*_{2a} \left\{ \bar{u}_1 P_L v_2 \times \frac{m_1 m_2^2}{(m_1^2 - m_2^2)} \right.$$

$$\times \left[-2\left(B_1^{(1)} + B_1^{(2)}\right) - \frac{1}{m_V^2}\left(m_1^2 B_1^{(1)} + m_2^2 B_1^{(1)}\right) \right.$$

$$\left. + \boxed{\mathbf{\frac{m_a^2}{m_V^2}\left(2(B_0^{(1)} - B_0^{(2)}) - (B_1^{(1)} + B_1^{(2)})\right)}} \right]$$

$$+ \bar{u}_1 P_R v_2 \frac{m_1^2 m_2}{m_1^2 - m_2^2} \left[(2-d)\left(B_1^{(1)} + B_1^{(2)}\right) - \frac{1}{m_V^2}\left(m_1^2 B_1^{(1)} + m_2^2 B_1^{(1)}\right) \right.$$

$$\left. \left. + \boxed{\mathbf{\frac{m_a^2}{m_V^2}\left(2\left(B_0^{(1)} - B_0^{(2)}\right) - \left(B_1^{(1)} + B_1^{(2)}\right)\right)}} \right] \right\}. \tag{B.18}$$

We note that the divergence part in the above expression is zero. The final result is

$$\mathcal{M}^{FV}_{(g+h)} = \left[\frac{1}{64\pi^2\sqrt{2}} \times \frac{g^3 c_\alpha}{s_\theta}\right] \sum_a V_{1a}V^*_{2a} \left[(\overline{u_1} P_L v_2)\, E_L^{FV} + (\overline{u_1} P_R v_2)\, E_R^{FV}\right], \tag{B.19}$$

where $E^{FV}_{L,R}$ are defined in (16) and (17).

The contribution from the diagram 1i) is:

$$i\mathcal{M}^{FH}_{(i)} = \sum_a \int \frac{d^4k}{(2\pi)^4} \times \bar{u}_1(-i\sqrt{2}V_{1a}) \left(\frac{m_1}{v_1} a_1 P_L + \frac{m_a}{v_3} a_3 P_R\right) \frac{i(\not{k} + m_a)}{D_0}$$

$$\times (-i\sqrt{2}V^*_{2a}) \left(\frac{m_2}{v_1} a_1 P_R + \frac{m_a}{v_3} a_3 P_L\right) \frac{i(\not{p}_1 + m_2)}{p_1^2 - m_2^2} \left(\frac{i m_2}{v_1}\frac{c_\alpha}{\sqrt{2}}\right) v_2 \times \frac{i}{D_1}$$

$$= \sum_a \left[-\frac{\sqrt{2}c_\alpha}{v_1}\right] \frac{m_2}{m_1^2 - m_2^2} V_{1a}V^*_{2a}$$

$$\times \int \frac{d^4k}{(2\pi)^4} \times \left[\frac{m_1 m_2}{v_1^2} a_1^2 \frac{\bar{u}_1 \not{k}\not{p}_1 P_L v_2}{D_0 D_1} + \frac{m_1 m_2^2}{v_1^2} a_1^2 \frac{\bar{u}_1 \not{k} P_R v_2}{D_0 D_1} \right.$$

$$+ \frac{m_1 m_a^2}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 \not{p}_1 P_R v_2}{D_0 D_1} + \frac{m_1 m_2 m_a^2}{v_1 v_3} a_1 a_3 \frac{\bar{u}_1 P_L v_2}{D_0 D_1} + \frac{m_a^2}{v_3^2} a_3^2 \frac{\bar{u}_1 \not{k}\not{p}_1 P_R v_2}{D_0 D_1}$$

$$+\frac{m_2 m_a^2}{v_3^2}a_3^2\frac{\bar{u}_1\not{k}P_L v_2}{D_0 D_1}+\frac{m_2 m_a^2}{v_1 v_3}a_1 a_3\frac{\bar{u}_1\not{p}_1 P_L v_2}{D_0 D_1}+\frac{m_2^2 m_a^2}{v_1 v_3}a_1 a_3\frac{\bar{u}_1 P_R v_2}{D_0 D_1}\Bigg]$$

$$=\sum_a\left[-\frac{\sqrt{2}c_\alpha}{v_1}\right]\frac{m_2}{m_1^2-m_2^2}V_{1a}V_{2a}^*$$

$$\times\Bigg\{\bar{u}_1 P_L v_2\times m_1 m_2\left[\boxed{\mathbf{2m_a^2\frac{a_1a_3}{v_1v_3}B_0^{(1)}+m_a^2\frac{a_3^2}{v_3^2}B_1^{(1)}}}+\frac{m_1^2}{v_1^2}a_1^2 B_1^{(1)}\right]$$

$$+\bar{u}_1 P_R v_2\left[\boxed{\mathbf{m_a^2\frac{a_1a_3}{v_1v_3}(m_1^2+m_2^2)B_0^{(1)}+m_1^2m_a^2\frac{a_3^2}{v_3^2}B_1^{(1)}}}+\frac{m_1^2m_2^2}{v_1^2}a_1^2 B_1^{(1)}\right]\Bigg\}. \tag{B.20}$$

The contribution from the diagram 1k) is:

$$i\mathcal{M}_{(k)}^{HF}=\sum_a\int\frac{d^4k}{(2\pi)^4}\times\bar{u}_1\left(\frac{im_1c_\alpha}{v_1\sqrt{2}}\right)$$

$$\times\frac{i(-\not{p}_2+m_1)}{p_2^2-m_1^2}(-i\sqrt{2}V_{1a})\left(\frac{m_1}{v_1}a_1P_L+\frac{m_a}{v_3}a_3P_R\right)$$

$$\times\frac{i(\not{k}+m_a)}{D_0}(-i\sqrt{2}V_{2a}^*)\left(\frac{m_2}{v_1}a_1P_R+\frac{m_a}{v_3}a_3P_L\right)v_2\times\frac{i}{D_2}$$

$$=\sum_a\left(-\frac{i\sqrt{2}c_\alpha}{v_1}\right)\frac{m_1}{m_2^2-m_1^2}V_{1a}V_{2a}^*$$

$$\times\int\frac{d^4k}{(2\pi)^4}\times\Bigg[-\frac{m_1m_2}{v_1^2}a_1^2\frac{\bar{u}_1\not{p}_2\not{k}P_Rv_2}{D_0D_2}+\frac{m_1^2m_2}{v_1^2}a_1^2\frac{\bar{u}_1\not{k}P_Rv_2}{D_0D_2}$$

$$-\frac{m_1m_a^2}{v_1v_3}a_1a_3\frac{\bar{u}_1\not{p}_2P_Lv_2}{D_0D_2}+\frac{m_1^2m_a^2}{v_1v_3}a_1a_3\frac{\bar{u}_1P_Lv_2}{D_0D_2}-\frac{m_a^2}{v_3^2}a_3^2\frac{\bar{u}_1\not{p}_2\not{k}P_Lv_2}{D_0D_2}$$

$$+\frac{m_1m_a^2}{v_3^2}a_3^2\frac{\bar{u}_1\not{k}P_Lv_2}{D_0D_2}-\frac{m_2m_a^2}{v_1v_3}a_1a_3\frac{\bar{u}_1\not{p}_2P_Rv_2}{D_0D_1}+\frac{m_1m_2m_a^2}{v_1v_3}a_1a_3\frac{\bar{u}_1P_Rv_2}{D_0D_1}\Bigg]$$

$$=\sum_a\left(-\frac{i\sqrt{2}c_\alpha}{v_1}\right)\frac{m_1}{m_2^2-m_1^2}V_{1a}V_{2a}^*\Bigg\{\bar{u}_1P_Lv_2\Bigg[\boxed{\mathbf{\frac{m_a^2}{v_1v_3}a_1a_3(m_1^2+m_2^2)B_0^{(2)}}}$$

$$-\boxed{\mathbf{\frac{m_2^2m_a^2}{v_3^2}a_3^2B_1^{(2)}}}-\frac{m_1^2m_2^2}{v_1^2}a_1^2B_1^{(2)}\Bigg]$$

$$+\bar{u}_1P_Rv_2\times m_1m_2\left[\boxed{\mathbf{2\frac{m_a^2}{v_1v_3}a_1a_3B_0^{(2)}-\frac{m_a^2}{v_3^2}a_3^2B_1^{(2)}}}-\frac{m_2^2}{v_1^2}a_1^2B_1^{(2)}\right]\Bigg\}. \tag{B.21}$$

The total amplitude from the two diagrams 1i) and k) is:

$$
\begin{aligned}
i\mathcal{M}^{FH}_{(i+k)} =& \sum_a V_{1a}V^*_{2a}\left[-\frac{i\sqrt{2}c_\alpha}{v_1}\right]\times\left\{\bar{u}_1P_Lv_2\times\frac{m_1}{m_1^2-m_2^2}\right.\\
&\times\left[m_1^2m_2^2\frac{a_1^2}{v_1^2}\left(B_1^{(1)}+B_1^{(2)}\right)+\boxed{\mathbf{m_a^2\frac{a_1a_3}{v_1v_3}\left(2m_2^2B_0^{(1)}-(m_1^2+m_2^2)B_0^{(2)}\right)}}\right.\\
&\left.+\boxed{\mathbf{m_2^2m_a^2\frac{a_3^2}{v_3^2}\left(B_1^{(1)}+B_1^{(2)}\right)}}\right]+\bar{u}_1P_Rv_2\times\frac{m_2}{m_1^2-m_2^2}\\
&\times\left[m_1^2m_2^2\frac{a_1^2}{v_1^2}\left(B_1^{(1)}+B_1^{(2)}\right)+\boxed{\mathbf{m_1^2m_a^2\frac{a_3^2}{v_3^2}\left(B_1^{(1)}+B_1^{(2)}\right)}}\right.\\
&\left.\left.+\boxed{\mathbf{m_a^2\frac{a_1a_3}{v_1v_3}\left(-2m_1^2B_0^{(2)}+(m_1^2+m_2^2)B_0^{(1)}\right)}}\right]\right\}. \qquad \text{(B.22)}
\end{aligned}
$$

The final result is written as

$$\mathcal{M}^{FH}_{(ik)} = \left[-\frac{8c_\alpha}{64\pi^2\sqrt{2}}\right]\sum_a V_{1a}V^*_{2a}\left[(\overline{u_1}P_Lv_2)\,E^{FH}_L+(\overline{u_1}P_Rv_2)\,E^{FH}_R\right], \qquad \text{(B.23)}$$

where $E^{FH}_{L,R}$ are defined in (18) and (19). After calculating contributions from all diagrams with virtual neutral leptons N_a we can prove that all divergent parts containing the factor m_a^2 will be canceled in the total contribution. The details are shown below. For active neutrinos the calculation is the same.

B.2. Particular calculation for canceling divergence

In this section, for contribution of exotic neutral leptons N_a we use the following relations

$$
\begin{aligned}
&a_1 \to c_\theta,\ a_2\to a_3 = s_\theta,\ v_1 = \frac{2m_V}{g}s_\theta,\ v_3=\frac{2m_V}{g}c_\theta,\\
&\frac{a_1}{v_1} = \frac{g}{2m_V}\frac{c_\theta}{s_\theta},\ \frac{a_3}{v_3}=\frac{g}{2m_V}\frac{s_\theta}{c_\theta},\ \frac{a_1a_3}{v_1v_3}=\frac{g^2}{4m_V^2}. \qquad \text{(B.24)}
\end{aligned}
$$

And we concentrate on the divergent parts which are bolded in the expressions of the amplitudes calculated above. With the notations of the divergences shown in the Appendix A, all of divergent parts are collected as follows,

$$
\begin{aligned}
&\text{Div}\left[\mathcal{M}^{FVV}_{(a)}\right] = B\times\left[c_\alpha\times(-3s_\theta)+\sqrt{2}s_\alpha(3c_\theta)\right],\\
&\text{Div}\left[\mathcal{M}^{FHV}_{(b+c)}\right] = B\times\left[c_\alpha\times\frac{s_\theta^2-2c_\theta^2}{s_\theta}+\sqrt{2}s_\alpha\times\frac{s_\theta^2-2c_\theta^2}{c_\theta}\right],\\
&\text{Div}\left[\mathcal{M}^{VFF}_{(e)}\right] = B\times\sqrt{2}s_\alpha\times\frac{-3}{c_\theta},
\end{aligned}
$$

$$\mathrm{Div}\left[\mathcal{M}^{HFF}_{(f)}\right] = B \times \sqrt{2}s_{\alpha} \times \frac{2}{c_{\theta}},$$
$$\mathrm{Div}\left[\mathcal{M}^{FV}_{(g)}\right] = \frac{1}{m_1^2-m_2^2}\left[m_2^2 B_L + m_1^2 B_R\right] \times \frac{3c_{\alpha}}{s_{\theta}},$$
$$\mathrm{Div}\left[\mathcal{M}^{FV}_{(h)}\right] = \frac{1}{m_1^2-m_2^2}\left[m_2^2 B_L + m_1^2 B_R\right] \times \frac{-3c_{\alpha}}{s_{\theta}},$$
$$\mathrm{Div}\left[\mathcal{M}^{FH}_{(i+k)}\right] = B \times c_{\alpha} \times \frac{2}{s_{\theta}}, \quad \text{(B.25)}$$

where

$$B = \frac{g^3}{128\pi^2}\frac{m^2_{\nu_a}}{m^3_W} \times \Delta_{\epsilon} \times [\bar{u}_1 P_L v_2 \times m_1 + \bar{u}_1 P_R v_2 \times m_2]$$
$$B_L = \frac{g^3}{128\pi^2}\frac{m^2_{\nu_a}}{m^3_W} \times \Delta_{\epsilon} \times \bar{u}_1 P_L v_2 \times m_1,\ B_R = \frac{g^3}{128\pi^2}\frac{m^2_{\nu_a}}{m^3_W} \times \Delta_{\epsilon} \times \bar{u}_1 P_R v_2 \times m_2.$$

It is easy to see that the sum over all factors is zero. Furthermore, it is interesting to see that the sums of the two parts having factor c_{α} and $\sqrt{2}s_{\alpha}$ independently result the zero values. From (41), the factor c_{α} arises from the contributions of neutral components of η and ρ, while the s_{α} factor arises from the contribution of χ.

For contribution of the active neutrinos, the two diagrams (b) and (c) of the Fig. 1 do not give contributions due to absence of the $H_2^- H_2^+ W$ couplings. Using the following properties

$$a_1 = 1, a_2 = 1, v_1 = v_2 = \frac{2m_W}{\sqrt{2}g},\ \frac{a_1}{v_1} = \frac{a_2}{v_2} = \frac{\sqrt{2}g}{2m_W},\ \frac{a_1 a_2}{v_1 v_2} = \frac{g^2}{2m_W^2},$$

we list the non-zero divergent terms of the relevant diagrams as follows

$$\mathrm{Div}\left[\mathcal{M}^{FVV}_{(a)}\right] = \mathcal{B} \times (-3c_{\alpha}),$$
$$\mathrm{Div}\left[\mathcal{M}^{VFF}_{(e)}\right] = \mathcal{B} \times (3c_{\alpha}),$$
$$\mathrm{Div}\left[\mathcal{M}^{HFF}_{(f)}\right] = \mathcal{B} \times (-2c_{\alpha}),$$
$$\mathrm{Div}\left[\mathcal{M}^{FV}_{(g)}\right] = \frac{1}{m_1^2-m_2^2}\left[m_2^2\mathcal{B}'_L + m_1^2\mathcal{B}'_R\right] \times (c_{\alpha}),$$
$$\mathrm{Div}\left[\mathcal{M}^{FV}_{(h)}\right] = \frac{1}{m_1^2-m_2^2}\left[m_2^2\mathcal{B}'_L + m_1^2\mathcal{B}'_R\right] \times (-c_{\alpha}),$$
$$\mathrm{Div}\left[\mathcal{M}^{FH}_{(i)}\right] = \frac{-c_{\alpha}}{m_1^2-m_2^2}\left[5m_2^2\mathcal{B}'_L + (3m_1^2+2m_2^2)\mathcal{B}'_R\right],$$
$$\mathrm{Div}\left[\mathcal{M}^{FH}_{(k)}\right] = \frac{c_{\alpha}}{m_1^2-m_2^2}\left[(2m_1^2+3m_2^2)\mathcal{B}'_L + 5m_1^2\mathcal{B}'_R\right],$$
$$\mathrm{Div}\left[\mathcal{M}^{FH}_{(i+k)}\right] = \mathcal{B} \times (2c_{\alpha}),$$

where

$$\mathcal{B} = \frac{g^3}{128\pi^2}\frac{m_{\nu_a}^2}{m_W^3} \times \Delta_\epsilon \times [\bar{u}_1 P_L v_2 \times m_1 + \bar{u}_1 P_R v_2 \times m_2]$$

$$\mathcal{B}'_L = \frac{g^3}{128\pi^2}\frac{m_{\nu_a}^2}{m_W^3} \times \Delta_\epsilon \times \bar{u}_1 P_L v_2 \times m_1, \ \mathcal{B}'_R = \frac{g^3}{128\pi^2}\frac{m_{\nu_a}^2}{m_W^3} \times \Delta_\epsilon \times \bar{u}_1 P_R v_2 \times m_2.$$

We see again that sum of all divergent terms is zero.

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