Chapter 9
Anomalies

So far we did not worry about how the classical symmetries of a theory are carried over to the quantum theory. We have implicitly assumed that classical symmetries are preserved in the process of quantization.

This is not necessarily the case. As we have seen in the previous chapter, quantizing an interacting field theory is a very involved process requiring regularization and renormalization. Sometimes, it does not matter how hard we try, there is no way for a classical symmetry to survive quantization. When this happens one says that the theory has an *anomaly* (for a review see [1]). It is important to avoid the misconception that anomalies appear due to a bad choice of the way a theory is regularized in the process of quantization. When we talk about anomalies we mean a classical symmetry that *cannot* be realized in the quantum theory, no matter how smart we are in choosing the regularization procedure.

In Chap. 8 we have already encountered an example of an anomaly: the quantum breaking of classical scale invariance reflected in the running of the coupling constants with the energy. In the following we focus on other examples of anomalies, this time associated with the global and local symmetries of the classical theory.

9.1 A Toy Model for the Axial Anomaly

Probably the best known examples of anomalies appear when we consider axial symmetries. In a theory of two Weyl spinors $u_\pm$

$$\mathscr{L} = i\overline{\psi}\partial\!\!\!/\psi = iu_+^\dagger\sigma_+^\mu\partial_\mu u_+ + iu_-^\dagger\sigma_-^\mu\partial_\mu u_- \quad \text{with} \quad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{9.1}$$

the Lagrangian is invariant under two types of global U(1) transformations. In the first one both chiralities transform with the same phase, this is a *vector* transformation:

$$\mathrm{U}(1)_\mathrm{V} : u_\pm \longrightarrow e^{i\alpha}u_\pm, \tag{9.2}$$

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whereas in the second, the axial U(1), the signs of the phases are different for the two chiralities

$$\mathrm{U}(1)_{\mathrm{A}} : u_{\pm} \longrightarrow e^{\pm i\alpha} u_{\pm}. \tag{9.3}$$

Using Noether's theorem, there are two conserved currents, a vector current

$$J^{\mu}_{\mathrm{V}} = \overline{\psi}\gamma^{\mu}\psi = u^{\dagger}_{+}\sigma^{\mu}_{+}u_{+} + u^{\dagger}_{-}\sigma^{\mu}_{-}u_{-} \qquad \Longrightarrow \qquad \partial_{\mu}J^{\mu}_{\mathrm{V}} = 0 \tag{9.4}$$

and an axial vector current

$$J^{\mu}_{\mathrm{A}} = \overline{\psi}\gamma^{\mu}\gamma_5\psi = u^{\dagger}_{+}\sigma^{\mu}_{+}u_{+} - u^{\dagger}_{-}\sigma^{\mu}_{-}u_{-} \qquad \Longrightarrow \qquad \partial_{\mu}J^{\mu}_{\mathrm{A}} = 0. \tag{9.5}$$

The theory described by the Lagrangian (9.1) can be coupled to the electromagnetic field. The resulting classical theory is still invariant under the vector and axial U(1) symmetries (9.2) and (9.3). Surprisingly, upon quantization it turns out that the conservation of the axial vector current (9.5) is spoiled by quantum effects

$$\partial_{\mu}J^{\mu}_{\mathrm{A}} \sim \hbar\,\mathbf{E}\cdot\mathbf{B}. \tag{9.6}$$

To understand more clearly how this result comes about, we study first a simple model in two dimensions that captures the relevant physics involved in the four-dimensional case [2]. We work in a two-dimensional Minkowski space with coordinates $(x^0, x^1) \equiv (t, x)$ and where the spatial direction is compactified to a circle S^1 with length L. In this setup we consider a fermion coupled to a classical electromagnetic field. Notice that in our two-dimensional world the field strength $\mathscr{F}_{\mu\nu}$ has only one independent component that corresponds to the electric field, $\mathscr{F}_{01} \equiv -\mathscr{E}$ (in two dimensions there are no magnetic fields!).

To write the Lagrangian for the spinor field we need to find a representation of the algebra of γ-matrices

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \quad \text{with} \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.7}$$

In two dimensions the dimension of the representation of the γ-matrices is 2. In fact, remembering the anticommutation relation of the Pauli matrices $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$ is not very difficult to come up with the following representation

$$\gamma^0 \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 \equiv i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9.8}$$

This is a chiral representation since the matrix γ_5 is diagonal[1]

$$\gamma_5 \equiv -\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.9}$$

[1] In any even number of dimensions γ_5 is defined to satisfy the conditions $(\gamma_5)^2 = \mathbf{1}$ and $\{\gamma_5, \gamma^{\mu}\} = 0$.

Writing a two-component Dirac spinor ψ as

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{9.10}$$

and defining as usual the projectors $P_\pm = \frac{1}{2}(\mathbf{1} \pm \gamma_5)$, we find that the components $u_\pm$ of ψ are respectively right- and left-handed Weyl spinors in two dimensions.

Once we have a representation of the γ-matrices we can write the Dirac equation. Expressed in terms of the components $u_\pm$ of the Dirac spinor, we have

$$(\partial_0 - \partial_1)u_+ = 0, \quad (\partial_0 + \partial_1)u_- = 0. \tag{9.11}$$

The general solution of these equations can be immediately written as

$$u_+ = u_+(x^0 + x^1), \quad u_- = u_-(x^0 - x^1). \tag{9.12}$$

Hence $u_\pm$ are two wave packets moving along the spatial dimension respectively to the left (u_+) and to the right (u_-). Notice that according to our convention the left-moving u_+ is a right-handed spinor (positive helicity) whereas the right-moving u_- is a left-handed spinor (negative helicity).

If we insist in interpreting (9.11) as the wave equation for two-dimensional Weyl spinors, we find the following properly normalized wave functions for free particles with well defined energy-momentum $p^\mu = (E, p)$

$$v_\pm^{(E)}(x^0 \pm x^1) = \frac{1}{\sqrt{L}} e^{-iE(x^0 \pm x^1)} \quad \text{with} \quad p = \mp E. \tag{9.13}$$

As it is always the case with a relativistic wave equation, we have found both positive and negative energy solutions. For $v_+^{(E)}$, since $E = -p$, we see that the solutions with positive energy are those with negative momentum $p < 0$, whereas the negative energy solutions are plane waves with $p > 0$. For the left-handed spinor u_- the situation is reversed. Besides, since the spatial direction is compact with length L the momentum p is quantized according to

$$p = \frac{2\pi n}{L}, \quad n \in \mathbb{Z}. \tag{9.14}$$

The spectrum of the theory is represented in Fig. 9.1.

Knowing the spectrum of the theory the next step is to obtain the vacuum. As with the Dirac equation in four dimensions, we identify the ground state of the theory with the one where all states with $E \leqslant 0$ are filled (see Fig. 9.2.). Exciting a particle in the Dirac sea produces a positive energy fermion plus a hole that is interpreted as an antiparticle. This gives us the key on how to quantize the theory. In the expansion of the operator $u_\pm$ in terms of the modes (9.13) we associate positive energy states with annihilation operators, whereas the states with negative energy are associated with creation operators for the corresponding antiparticle

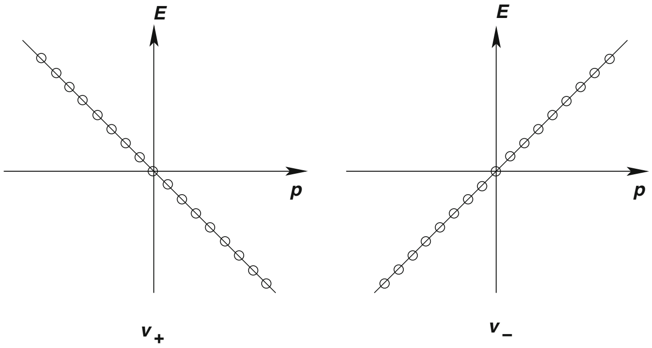


Fig. 9.1 Spectrum of the massless two-dimensional Dirac field. We denote by $v_\pm$ the states with dispersion relation $E = \mp p$

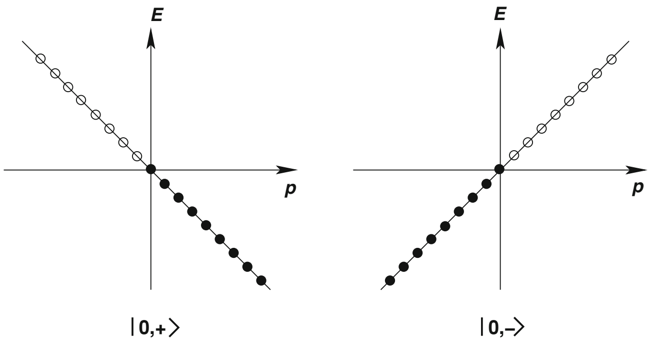


Fig. 9.2 The two branches in the vacuum of the theory. The solid points represent the filled negative energy states

$$u_\pm(x) = \sum_{E>0} \left[a_\pm(E) v_\pm^{(E)}(x) + b_\pm^\dagger(E) v_\pm^{(E)}(x)^* \right]. \tag{9.15}$$

The operator $a_\pm(E)$ annihilates a particle with positive energy E and momentum $\mp E$, and $b_\pm^\dagger(E)$ creates out of the vacuum an antiparticle with positive energy E and spatial momentum $\mp E$. In the Dirac sea picture the operator $b_\pm(E)^\dagger$ is originally an annihilation operator for a state of the sea with negative energy $-E$. As in four dimensions, the problem of the negative energy states is solved by interpreting annihilation operators for negative energy states as creation operators for the corresponding antiparticle with positive energy (and vice versa). The operators appearing in the expansion of $u_\pm$ in Eq. (9.15) satisfy the usual fermionic algebra

$$\{a_\lambda(E), a_{\lambda'}^\dagger(E')\} = \{b_\lambda(E), b_{\lambda'}^\dagger(E')\} = \delta_{E,E'}\delta_{\lambda\lambda'}, \tag{9.16}$$

where we have introduced the label $\lambda, \lambda' = \pm$. In addition, $a_\lambda(E),\ a_\lambda^\dagger(E)$ anticommute with $b_{\lambda'}(E'),\ b_{\lambda'}^\dagger(E')$.

The Lagrangian of the theory

$$\mathscr{L} = iu_+^\dagger(\partial_0 + \partial_1)u_+ + iu_-^\dagger(\partial_0 - \partial_1)u_- \tag{9.17}$$

is invariant under both the $U(1)_V$ transformations shown in Eq. (9.2), and $U(1)_A$ of Eq. (9.3). The corresponding Noether currents are

$$J_V^\mu = \begin{pmatrix} u_+^\dagger u_+ + u_-^\dagger u_- \\ -u_+^\dagger u_+ + u_-^\dagger u_- \end{pmatrix}, \quad J_A^\mu = \begin{pmatrix} u_+^\dagger u_+ - u_-^\dagger u_- \\ -u_+^\dagger u_+ - u_-^\dagger u_- \end{pmatrix}. \tag{9.18}$$

The associated conserved charges are given by

$$Q_V \equiv \int_0^L dx^1 J_V^0 = \int_0^L dx^1 \left(u_+^\dagger u_+ + u_-^\dagger u_-\right), \tag{9.19}$$

for the vector current, and

$$Q_A \equiv \int_0^L dx^1 J_A^0 = \int_0^L dx^1 \left(u_+^\dagger u_+ - u_-^\dagger u_-\right) \tag{9.20}$$

for the vector axial one. Using the orthonormality relations for the modes $v_\pm^{(E)}(x)$

$$\int_0^L dx^1 v_\pm^{(E)}(x)^* v_\pm^{(E')}(x) = \delta_{E,E'}, \tag{9.21}$$

the conserved charges can be explicitly computed as

$$\begin{aligned} Q_V &= \sum_{E>0}\Big[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) + a_-^\dagger(E)a_-(E) - b_-^\dagger(E)b_-(E)\Big], \\ Q_A &= \sum_{E>0}\Big[a_+^\dagger(E)a_+(E) - b_+^\dagger(E)b_+(E) - a_-^\dagger(E)a_-(E) + b_-^\dagger(E)b_-(E)\Big]. \end{aligned} \tag{9.22}$$

From these expressions we see how Q_V counts the net fermion number, i.e. the number of particles minus antiparticles, independently of their helicity. The axial charge Q_A, on the other hand, counts the net number of positive minus negative helicity states. In the case of the vector current we have subtracted a formally divergent vacuum contribution to the charge (the "charge of the Dirac sea").

In the free theory there is of course no problem with the conservation of either Q_V or Q_A, since the occupation numbers do not change. What we want to study is the effect of coupling the theory to the electric field $\mathscr{E}$. We work in the gauge $\mathscr{A}^0 = 0$. Instead of solving the problem exactly we are going to use the following trick: we simulate the electric field by adiabatically varying in a long time τ_0 the

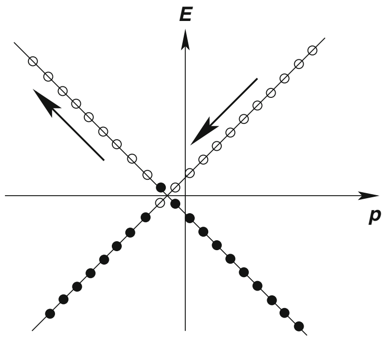


Fig. 9.3 Effect of the electric field on the vacuum shown in Fig. 9.2. Some of the occupied negative energy states in the brach v_+ acquires positive energy, while the same number of empty positive energy states in the branch v_- shift to negative energy and become holes in the Dirac sea

vector potential $\mathscr{A}^1$ from zero value to $\mathscr{E}\tau_0$. From our discussion in Chap. 4 (see Sect. 4.1) we know that the effect of the electromagnetic coupling in the theory is a shift in the momentum according to

$$p \longrightarrow p - e\mathscr{A}^1, \tag{9.23}$$

where e is the charge of the fermions. Since we assumed that the vector potential varies adiabatically, we can take it to be approximately constant at each time.

We have to understand the effect on the vacuum depicted in Fig. 9.2 of switching on the vector potential. Increasing adiabatically $\mathscr{A}^1$ results, according to Eq. (9.23), in decreasing the momentum of the state. What happens to the energy depends on whether we consider states with dispersion relation $E = -p$ (the branch v_+) or $E = p$ (the branch v_-).

The result is that the two branches move as shown in Fig. 9.3. Thus, some of the negative energy states of the v_+ branch acquire positive energy while the same number of the empty positive energy states of the other branch v_- become empty negative energy states. Physically, this means that the external electric field $\mathscr{E}$ creates a number of particle-antiparticle pairs out of the vacuum.

We have to count the number of such pairs created by the electric field after a time τ_0. This is given by

$$N = \frac{L}{2\pi} e\mathscr{E}\tau_0. \tag{9.24}$$

To get this expression we have divided the shift of the spectrum $e\mathscr{E}\tau_0$ by the separation between energy levels given by $\frac{2\pi}{L}$ [cf. Eq. (9.14)]. The value of the charges at the time τ_0 are

$$\begin{aligned} Q_V(\tau_0) &= (N - 0) + (0 - N) = 0, \\ Q_A(\tau_0) &= (N - 0) - (0 - N) = 2N. \end{aligned} \tag{9.25}$$

We conclude that the coupling to the electric field produces a violation in the conservation of the axial charge per unit time given by

$$\dot{Q}_\mathrm{A} = \frac{e}{\pi}\mathscr{E}L. \tag{9.26}$$

This result translates into a nonconservation of the axial vector current

$$\partial_\mu J^\mu_\mathrm{A} = \frac{e\hbar}{\pi}\mathscr{E}, \tag{9.27}$$

where we have restored $\hbar$ to make clear that we are dealing with a quantum effect. In addition, the fact that $\Delta Q_\mathrm{V} = 0$ guarantees that the vector current remains conserved also quantum mechanically, $\partial_\mu J^\mu_\mathrm{V} = 0$.

9.2 The Triangle Diagram

We have just studied a two-dimensional example of the Adler-Bell-Jackiw axial anomaly [3, 4]. We have presented a heuristic analysis consisting of studying the coupling of a two-dimensional massless fermion to an external classical electric field to compute the violation in the conservation of the axial vector current due to quantum effects.

This suggests an alternative, more sophisticated way to compute the axial anomaly. Gauge invariance requires that the fermion couples to the external gauge field through the vector current J^μ_V via a term in the Lagrangian

$$\mathscr{L} = i\overline{\psi}\partial\!\!\!/\psi + eJ^\mu_\mathrm{V}\mathscr{A}_\mu, \tag{9.28}$$

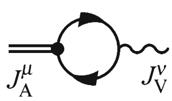
where $\mathscr{A}_\mu(x)$ represents the classical external gauge field. To decide whether the axial vector current is conserved quantum mechanically we compute the vacuum expectation value

$$\left\langle \partial_\mu J^\mu_\mathrm{A}(x)\right\rangle_\mathscr{A}, \tag{9.29}$$

where the subscript indicates the expectation value is computed in the vacuum of the theory coupled to the external field. This quantity can be evaluated in powers of $\mathscr{A}_\mu$ using either the operator formalism or functional integrals. The first nonvanishing term is

$$\left\langle \partial_\mu J^\mu_\mathrm{A}(x)\right\rangle_\mathscr{A} = ie\int d^2y\,\partial_\mu C^{\mu\nu}(y)\mathscr{A}_\nu(x-y), \tag{9.30}$$

where

$$C^{\mu\nu}(x) = \langle 0|T\left[J^\mu_\mathrm{A}(x)J^\nu_\mathrm{V}(0)\right]|0\rangle =$$ 

(9.31)

In this correlation function the state $|0\rangle$ represents the Fock space vacuum of the free fermion theory. It can be evaluated using Wick's theorem. The Feynman diagram summarizes the Wick contractions required to compute the time-ordered correlation function of the two currents

$$C^{\mu\nu}(x) = \langle 0|\overline{\psi}\gamma^{\mu}\gamma_5\psi(x)\,\overline{\psi}\gamma^{\nu}\psi(0)|0\rangle. \tag{9.32}$$

We have concluded that the axial anomaly is controlled by the quantity $\partial_\mu C^{\mu\nu}(x)$. In computing the anomaly we have to impose the conservation of the vector current. This is crucial, since the gauge invariance of the theory depends upon it.[2] Doing this, one arrives at the result

$$\langle \partial_\mu J^{\mu}_{\mathrm{A}}\rangle_{\mathscr{A}} = -\frac{e\hbar}{2\pi}\varepsilon^{\nu\sigma}\mathscr{F}_{\nu\sigma}, \tag{9.33}$$

with $\varepsilon^{01} = -\varepsilon^{10} = 1$ and $\mathscr{F}_{\mu\nu}$ is the field strength of the external gauge field. It is immediate to check that the diagramatic calculation renders the same result (9.27) obtained in the previous section using a more heuristic argumentation.

The calculation of the axial anomaly can be also carried out in four dimensions along the same lines. Again, we have to compute the vacuum expectation value of the axial vector current coupled to an external classical gauge field $\mathscr{A}_\mu$. Now, however, the first nonvanishing contribution comes from the term quadratic in the external gauge field, namely

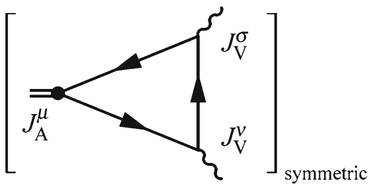
$$\langle \partial_\mu J^{\mu}\rangle_{\mathscr{A}} = -\frac{e^2}{2}\int d^4y_1 d^4y_2 \partial^{(x)}_{\mu} C^{\mu\nu\sigma}(x,y)\mathscr{A}_{\nu}(x-y_1+y_2)\mathscr{A}_{\sigma}(x-y_2), \tag{9.34}$$

where now

$$C^{\mu\nu\sigma}(x,y) = \big\langle 0|T\left[J^{\mu}_{\mathrm{A}}(x)J^{\nu}_{\mathrm{V}}(y)J^{\sigma}_{\mathrm{V}}(0)\right]|0\big\rangle. \tag{9.35}$$

This correlation function can be computed diagrammatically as

$C^{\mu\nu\sigma}(x,y) =$ (9.36)



[2] In fact there is a tension between the conservation of the vector an axial vector currents. The calculation of the diagram shown in Eq. (9.31) can be carried out imposing the conservation of the axial vector current, which results in an anomaly for the vector current. Since this would be disastrous for the consistency of the theory, we choose the other alternative.

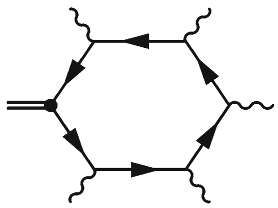
This is the celebrated triangle diagram. The subscript indicates that, in fact, $C^{\mu\nu\sigma}$ is given by two triangle diagrams with the two photon external legs interchanged. This is the result of Bose symmetry and can be explicitly checked by performing the Wick contractions in the correlation function (9.35).

The evaluation of the integral in the right-hand side of (9.34) is complicated by the presence of divergences that have to be regularized. As in the two-dimensional case, the conservation of the vector currents has to be imposed. The calculation gives the following anomaly for the axial vector current [3, 4]

$$\langle \partial_\mu J^\mu_{\rm A} \rangle_{\mathscr{A}} = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\sigma\lambda}\mathscr{F}_{\mu\nu}\mathscr{F}_{\sigma\lambda}. \tag{9.37}$$

This result has very important consequences in the physics of strong interactions as we will see in the next section.

We have paid attention to the axial anomaly in two and four dimensions. Chiral fermions exists in all even-dimensional space-times and, as a matter of fact, the axial vector current has an anomaly in all even-dimensional space-times. More precisely, if the dimension of the space-time is $d = 2k$, with $k = 1, 2, \ldots$, the anomaly is given by a one-loop diagram with one axial current and k vector currents, i.e. a $(k+1)$-gon. For example, in 10 dimensions the axial anomaly comes from the following hexagon diagram



As in the four-dimensional case, Bose symmetry and the conservation of all vector currents has to be imposed.

9.3 Chiral Symmetry in QCD

Our knowledge of the physics of strong interactions is based on the theory of Quantum Chromodynamics (QCD) introduced in Sect. 5.3 (see also [5–7] for reviews). Here we will consider a slightly more general version with an arbitrary number of colors and flavors: a nonabelian gauge theory with gauge group SU(N_c) coupled to a number N_f of quarks. These are spin-$\frac{1}{2}$ particles Q^f_i labelled by the color and flavor quantum numbers $i = 1, \ldots, N_c$ and $f = 1, \ldots, N_f$. The interaction between them is mediated by the $N_c^2 - 1$ gauge bosons, the gluons A^A_μ, with $A = 1, \ldots, N_c^2 - 1$. Let us recall that in the real world $N_c = 3$ and $N_f = 6$, corresponding to the six quarks: up (u), down (d), charm (c), strange (s), top (t) and bottom (b).

For reasons that will be clear later we work in the limit of vanishing quark masses[3] $m_f \to 0$. In this case the QCD Lagrangian is given by

$$\mathscr{L}_{\rm QCD} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} + \sum_{f=1}^{N_f} \Big(i \overline{Q}^f_L \not{D} Q^f_L + i \overline{Q}^f_R \not{D} Q^f_R \Big), \tag{9.38}$$

where the subscripts L and R indicate respectively left and right-handed spinors,

$$Q^f_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5) Q^f, \tag{9.39}$$

and the field strength $F^A_{\mu\nu}$ and covariant derivative D_μ are respectively defined in Eqs. (4.52) and (4.46). Apart from the gauge symmetry, this Lagrangian is also invariant under a global $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ acting on the flavor indices and defined by

$$\mathrm{U}(N_f)_L : \begin{cases} Q^f_L \to \sum_{f'=1}^{N_f} (U_L)_{ff'} Q^{f'}_L \\ Q^f_R \to Q^f_R \end{cases} \qquad \mathrm{U}(N_f)_R : \begin{cases} Q^f_L \to Q^f_L \\ Q^r_R \to \sum_{f'=1}^{N_f} (U_R)_{ff'} Q^{f'}_R \end{cases} \tag{9.40}$$

with $U_L, U_R \in \mathrm{U}(N_f)$. Since U(N) = U(1) × SU(N), this global symmetry group can be written as $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$. The abelian subgroup $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ can be now decomposed into their vector $\mathrm{U}(1)_B$ and axial $\mathrm{U}(1)_{\rm A}$ subgroups defined by the transformations

$$\mathrm{U}(1)_B : \begin{cases} Q^f_L \to e^{i\alpha} Q^f_L \\ Q^f_R \to e^{i\alpha} Q^f_R \end{cases} \qquad \mathrm{U}(1)_{\rm A} : \begin{cases} Q^f_L \to e^{i\alpha} Q^f_L \\ Q^f_R \to e^{-i\alpha} Q^f_R \end{cases}$$

According to Noether's theorem, associated with these two abelian symmetries we have two conserved currents:

$$J^\mu_{\rm V} = \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu Q^f, \quad J^\mu_{\rm A} = \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu \gamma_5 Q^f. \tag{9.41}$$

The conserved charge associated with the vector current $J^\mu_{\rm V}$ is the baryon number counting the number of quarks minus the number of antiquarks.

The nonabelian part of the global symmetry, group $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ can also be decomposed into its vector and axial factors, $\mathrm{SU}(N_f)_{\rm V} \times \mathrm{SU}(N_f)_{\rm A}$, defined by the following transformations of the quarks fields

[3] In the real world this makes sense only for the up and down, and perhaps the strange quarks.

$$\mathrm{SU}(N_f)_\mathrm{V}: \begin{cases} Q_L^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_L^{f'} \\ Q_R^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_R^{f'} \end{cases} \qquad \mathrm{SU}(N_f)_\mathrm{A}: \begin{cases} Q_L^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_L^{f'} \\ Q_R^f \to \sum_{f'=1}^{N_f} U_{ff'}^{-1} Q_R^{f'} \end{cases} \tag{9.42}$$

where U is a $\mathrm{SU}(N_f)$ matrix. Again, the application of Noether's theorem shows the existence of the following nonabelian conserved charges

$$J_\mathrm{V}^{I\mu} \equiv \sum_{f,f'=1}^{N_f} \overline{Q}^f \gamma^\mu (T^I)_{ff'} Q^{f'},$$
$$J_\mathrm{A}^{I\mu} \equiv \sum_{f,f'=1}^{N_f} \overline{Q}^f \gamma^\mu \gamma_5 (T^I)_{ff'} Q^{f'}. \tag{9.43}$$

To summarize, we have shown that the initial flavor chiral symmetry of the QCD Lagrangian (9.38) can be decomposed according to

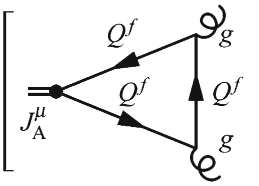
$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{SU}(N_f)_\mathrm{V} \times \mathrm{SU}(N_f)_\mathrm{A} \times \mathrm{U}(1)_B \times \mathrm{U}(1)_\mathrm{A}. \tag{9.44}$$

Up to now we have worked with the classical Lagrangian. The question to address next is which part of the classical global symmetry is preserved in the quantum theory.

As argued in Sect. 9.1, the conservation of the axial vector currents J_A^μ and $J_\mathrm{A}^{A\mu}$ can in principle be spoiled by an anomaly. In the case of the abelian axial current J_A^μ the relevant quantity to compute is the correlation function

$$C^{\mu\nu\sigma}(x,x') \equiv \langle 0|T\left[J_\mathrm{A}^\mu(x) j_\mathrm{gauge}^{A\nu}(x') j_\mathrm{gauge}^{B\sigma}(0)\right]|0\rangle$$

$$= \sum_{f=1}^{N_f} \Big[\quad \Big]_\mathrm{symmetric} \tag{9.45}$$



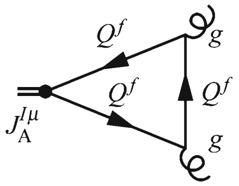
Here $j_\mathrm{gauge}^{A\mu}$ is the nonabelian conserved current coupling to the gluon field

$$j_\mathrm{gauge}^{A\mu} \equiv \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu \tau^A Q^f, \tag{9.46}$$

where, to avoid confusion with the generators of the global symmetry, we have denoted by τ^A the generators of the gauge group $\mathrm{SU}(N_c)$. The anomaly can be read now from $\partial_\mu^{(x)} C^{\mu\nu\sigma}(x, x')$. If we impose Bose symmetry with respect to the interchange of the two outgoing gluons and the conservation of the vector currents, we find that the axial abelian global current has an anomaly given by[4]

$$\partial_\mu J_{\text{A}}^\mu = -\frac{g^2 N_f}{32\pi^2} \varepsilon^{\mu\nu\sigma\lambda} F_{\mu\nu}^A F_{\sigma\lambda}^A. \tag{9.47}$$

In the case of the nonabelian axial global symmetry $\text{SU}(N_f)_{\text{A}}$ the calculation of the anomaly is made as above. The result, however, is quite different since in this case we conclude that the nonabelian axial vector current $J_{\text{A}}^{A\mu}$ is not anomalous. This can be easily seen by noticing that associated with the axial vector current vertex we have a generator T^I of $\text{SU}(N_f)$, whereas for the two gluon vertices we have the generators τ^A of the gauge group $\text{SU}(N_c.)$ Therefore, the triangle diagram is proportional to the group-theory factor

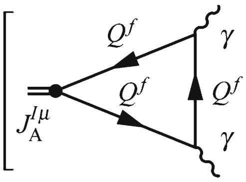


$$\left[\ \cdots\ \right]_{\text{symmetric}} \sim \text{Tr}\, T^I\, \text{Tr}\{\tau^A, \tau^B\} = 0 \tag{9.48}$$

vanishing because the generators of $\text{SU}(N_f)$ are traceless.

From here we could be tempted to conclude that the nonabelian axial symmetry $\text{SU}(N_f)_{\text{A}}$ is nonanomalous. However this is not the whole story, since quarks are charged particles that also couple to photons. Thus there is a second potential source of an anomaly coming from the the one-loop triangle diagram coupling $J_{\text{A}}^{I\mu}$ to two photons

$$\langle 0|T\left[J_{\text{A}}^{I\mu}(x) j_{\text{em}}^\nu(x') j_{\text{em}}^\sigma(0)\right]|0\rangle = \sum_{f=1}^{N_f} \left[\ \cdots\ \right]_{\text{symmetric}} \tag{9.49}$$



where j_{em}^μ is the electromagnetic current

$$j_{\text{em}}^\mu = \sum_{f=1}^{N_f} q_f \overline{Q}^f \gamma^\mu Q^f, \tag{9.50}$$

[4] The normalization of the generators T^I of the global $\text{SU}(N_f)$ is given by $\text{Tr}(T^I T^J) = \frac{1}{2}\delta^{IJ}$.

with q_f the electric charge of the f-th quark flavor. A calculation of the diagram in (9.49) shows the existence of the Adler-Bell-Jackiw anomaly given by

$$\partial_\mu J_{\text{A}}^{I\mu} = -\frac{N_c}{16\pi^2}\left[\sum_{f=1}^{N_f}(T^I)_{ff}q_f^2\right]\varepsilon^{\mu\nu\sigma\lambda}F_{\mu\nu}F_{\sigma\lambda}, \tag{9.51}$$

where $F_{\mu\nu}$ is the field strength of the electromagnetic field coupling to the quarks. The only chance for the anomaly to cancel is that the factor between brackets in this equation be identically zero.

Before proceeding let us summarize the results found so far. Due to the presence of anomalies the axial part of the global chiral symmetry, $\text{SU}(N_f)_{\text{A}}$ and $\text{U}(1)_{\text{A}}$, are not realized quantum mechanically in general. We found that $\text{U}(1)_{\text{A}}$ is always affected by an anomaly. However, the right-hand side of the anomaly equation (9.47) is a total derivative, so the anomalous character of J_{A}^{μ} does not explain the absence of $\text{U}(1)_{\text{A}}$ multiplets in the hadron spectrum, since a new current can be constructed which is conserved. In addition, the nonexistence of candidates for an associated Nambu-Goldstone boson with the right quantum numbers indicates that $\text{U}(1)_{\text{A}}$ is not spontaneously broken either, so it has to be explicitly broken somehow. This is the so-called U(1)-problem solved by 't Hooft [8], who showed how the contribution from instantons describing quantum transitions between vacua with topologically nontrivial gauge field configurations results in an explicit breaking of this symmetry.

Due to the dynamics of the $\text{SU}(N_c)$ gauge theory, the axial nonabelian symmetry is spontaneously broken due to the presence at low energies of a vacuum expectation value for the fermion bilinear $\overline{Q}^f Q^f$

$$\langle 0|\overline{Q}^f Q^f|0\rangle \neq 0 \qquad (\text{no summation in } f!). \tag{9.52}$$

This nonvanishing vacuum expectation value for the quark bilinear breaks chiral invariance spontaneously to the vector subgroup $\text{SU}(N_f)_{\text{V}}$, so the only subgroup of the original global symmetry that is realized in the full theory at low energy is

$$\text{U}(N_f)_L \times \text{U}(N_f)_R \longrightarrow \text{SU}(N_f)_{\text{V}} \times \text{U}(1)_B. \tag{9.53}$$

Associated with this breaking, Nambu–Goldstone bosons should appear with the quantum numbers of the broken nonabelian currents. For example, in the case of QCD the Nambu–Goldstone bosons associated with the spontaneous symmetry breaking induced by the vacuum expectation values $\langle\overline{u}u\rangle$, $\langle\overline{d}d\rangle$ and $\langle(\overline{u}d - \overline{d}u)\rangle$ have been identified as the pions π^0, $\pi^{\pm}$. These bosons are not exactly massless due to the nonvanishing mass of the u and d quarks. Since the global chiral symmetry is already slightly broken by mass terms in the Lagrangian, the associated Goldstone bosons also have masses although they are very light compared to the masses of other hadrons.

In order to have a better physical understanding of the role of anomalies in the physics of the strong interactions we particularize our analysis to the case of real QCD. Since the u and d quarks are much lighter than the other four flavors, QCD at low energies can be well described by including only these two flavors and ignoring heavier quarks. In this approximation, from our previous discussion we know that the low energy global symmetry of the theory is $\mathrm{SU}(2)_{\mathrm{V}} \times \mathrm{U}(1)_B$, where now the vector group $\mathrm{SU}(2)_{\mathrm{V}}$ is the well-known isospin symmetry. The axial $\mathrm{U}(1)_{\mathrm{A}}$ current is anomalous due to Eq. (9.47) with $N_f = 2$. In the case of the nonabelian axial symmetry $\mathrm{SU}(2)_{\mathrm{A}}$, taking into account that $q_u = \frac{2}{3}e$ and $q_d = -\frac{1}{3}e$ and that the three generators of SU(2) can be written in terms of the Pauli matrices as $T^K = \frac{1}{2}\sigma_K$ we find

$$\begin{aligned} \sum_{f=u,d} (T^1)_{ff} q_f^2 &= \sum_{f=u,d} (T^2)_{ff} q_f^2 = 0, \\ \sum_{f=u,d} (T^3)_{ff} q_f^2 &= \frac{e^2}{6}. \end{aligned} \tag{9.54}$$

Therefore $J_{\mathrm{A}}^{3\mu}$ is anomalous.

The anomaly in the axial vector current $J_{\mathrm{A}}^{3\mu}$ has an important physical consequence. As we learned in Chap. 5 the flavor wave function of the neutral pion π^0 is given by

$$|\pi^0\rangle = \frac{1}{\sqrt{2}} \left(|\bar{u}u\rangle - |\bar{d}d\rangle \right). \tag{9.55}$$

The isospin quantum numbers of $|\pi^0\rangle$ are those of $J_{\mathrm{A}}^{3\mu}$. In fact, the correspondence goes even further. The divergence of the axial vector current $\partial_\mu J_{\mathrm{A}}^{3\mu}$ has precisely the same quantum numbers as the pion. This means that, properly normalized, it can be identified as the operator creating a pion π^0 out of the vacuum

$$|\pi^0\rangle \sim \partial_\mu J_{\mathrm{A}}^{3\mu} |0\rangle. \tag{9.56}$$

This leads to the physical interpretation of the triangle diagram (9.49) with $J_{\mathrm{A}}^{3\mu}$ as the one loop contribution to the decay of a neutral pion into two photons

$$\pi^0 \longrightarrow 2\gamma. \tag{9.57}$$

This is an interesting piece of physics. In 1967 Sutherland and Veltman [9, 10] presented a calculation, using current algebra techniques, according to which the decay of the pion into two photons should be suppressed. This however contradicted the experimental evidence showing the existence of such a decay. The way out to this paradox, as pointed out in [3, 4], is the axial anomaly. What happens is that the current algebra analysis overlooks the ambiguities associated with the regularization of divergences in quantum field theory. A QED evaluation of the triangle diagram leads to a divergent integral that has to be regularized. It is in this process that the

Adler-Bell-Jackiw axial anomaly appears resulting in a nonvanishing value for the $\pi^0 \to 2\gamma$ amplitude.[5]

9.4 Gauge Anomalies

The existence of anomalies associated with global currents does not necessarily mean difficulties for the theory. On the contrary, as we saw in the case of the axial anomaly, its existence provides a solution of the Sutherland–Veltman paradox and an explanation of the electromagnetic decay of the pion. The situation is very different when we deal with local symmetries. A quantum mechanical violation of gauge symmetry leads to many problems, from lack of renormalizability to nondecoupling of negative norm states. This is because the presence of an anomaly in the theory implies that the Gauss' law constraint $\mathbf{D} \cdot \mathbf{E}_A = \rho_A$ cannot be consistently implemented in the quantum theory. As a consequence, states that classically were eliminated by the gauge symmetry become propagating in the quantum theory, thus spoiling the consistency of the theory.

Anomalies in a gauge symmetry can be expected only in chiral theories where left and right-handed fermions transform in different representations of the gauge group. Physically, the most interesting example of such theories is the electroweak sector of the standard model where, for example, left handed fermions transform as doublets under SU(2) whereas right-handed fermions are singlets. On the other hand, QCD is free of gauge anomalies since both left- and right-handed quarks transform in the fundamental representation of SU(3).

We consider the Lagrangian

$$\mathscr{L} = -\frac{1}{4} F^{A\mu\nu} F^A_{\mu\nu} + i \sum_{i=1}^{N_+} \overline{\psi}^i_+ \not{D}^{(+)} \psi^i_+ + i \sum_{j=1}^{N_-} \overline{\psi}^j_- \not{D}^{(-)} \psi^j_-, \tag{9.58}$$

where the chiral fermions $\psi^i_\pm$ transform according to the representations $\tau^A_{i,\pm}$ of the gauge group G ($A = 1, \ldots, \dim G$). The covariant derivatives $D^{(\pm)}_\mu$ are, as usual, defined by

$$D^{(\pm)}_\mu \psi^i_\pm = \partial_\mu \psi^i_\pm - i g_{\mathrm{YM}} A^A_\mu \tau^A_\pm \psi^i_\pm. \tag{9.59}$$

The anomaly is determined by the parity-violating part of the triangle diagram with three external gauge bosons, summed over all chiral fermion species running in the loop. All three vertices in the diagram include a projector P_+ or P_- and the parity-violating terms are identified as those containing a single γ_5. Splitting the gauge current into its vector and axial vector part, we conclude that the gauge anomaly comes from the triangle diagram with one axial and two vector gauge currents

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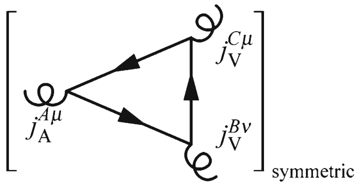
5 An early computation of the triangle diagram for the electromagnetic decay of the pion was made by Steinberger in [11].

$$\langle 0|T\left[j_{\rm A}^{A\mu}(x)j_{\rm V}^{B\nu}(x')j_{\rm V}^{C\sigma}(0)\right]|0\rangle = \tag{9.60}$$



where $j_{\rm V}^{A\mu}$ and $j_{\rm A}^{A\mu}$ are given by

$$\begin{aligned} j_{\rm V}^{A\mu} &= \sum_{i=1}^{N_+} \overline{\psi}_+^i \tau_+^A \gamma^\mu \psi_+^i + \sum_{j=1}^{N_-} \overline{\psi}_-^j \tau_-^A \gamma^\mu \psi_-^j, \\ j_{\rm A}^{A\mu} &= \sum_{i=1}^{N_+} \overline{\psi}_+^i \tau_+^A \gamma^\mu \psi_+^i - \sum_{i=1}^{N_-} \overline{\psi}_-^j \tau_-^A \gamma^\mu \psi_-^j. \end{aligned} \tag{9.61}$$

Luckily, we do not have to compute the whole diagram in order to find an anomaly cancellation condition. It is enough if we calculate the overall group theoretical factor. In the case of the diagram in Eq. (9.60) for each fermion species running in the loop this factor is equal to

$$\mathrm{Tr}\left[\tau_{i,\pm}^A\{\tau_{i,\pm}^B, \tau_{i,\pm}^C\}\right], \tag{9.62}$$

where the sign $\pm$ corresponds respectively to the generators of the representations of the gauge group for the left and right-handed fermions. Hence, the anomaly cancellation condition reads

$$\sum_{i=1}^{N_+} \mathrm{Tr}\left[\tau_{i,+}^A\{\tau_{i,+}^B, \tau_{i,+}^C\}\right] - \sum_{j=1}^{N_-} \mathrm{Tr}\left[\tau_{j,-}^A\{\tau_{j,-}^B, \tau_{j,-}^C\}\right] = 0. \tag{9.63}$$

Knowing this we can proceed to check the anomaly cancellation in the standard model SU(3)× SU(2)× U(1)$_Y$. Left handed fermions (both leptons and quarks) transform as doublets with respect to the SU(2) factor whereas the right-handed components are singlets. The charge with respect to the U(1)$_Y$ part, the weak hypercharge Y, is determined by the Gell-Mann–Nishijima formula

$$Q = T_3 + Y, \tag{9.64}$$

where Q is the electric charge of the corresponding particle and T_3 is the eigenvalue with respect to the third generator of the SU(2) group in the corresponding representation: $T_3 = \frac{1}{2}\sigma_3$ for the doublets and $T_3 = 0$ for the singlets. For the first family of quarks (u, d) and leptons (e, ν_e) we have the following field content

$$\begin{aligned} &\text{quarks:} \quad \begin{pmatrix} u^i \\ d^i \end{pmatrix}_{L,\frac{1}{6}} \qquad u^i_{R,\frac{2}{3}} \qquad d^i_{R,-\frac{1}{3}} \\ &\text{leptons:} \quad \begin{pmatrix} \nu_e \\ e \end{pmatrix}_{L,-\frac{1}{2}} \qquad e_{R,-1} \end{aligned} \tag{9.65}$$

where $i = 1, 2, 3$ labels the color quantum number and the subscript indicates the value of the weak hypercharge Y. Denoting the representations of SU(3)× SU(2) × U(1)$_Y$ by $(n_c, n_w)_Y$, with n_c and n_w the representations of SU(3) and SU(2) respectively and Y the hypercharge, the matter content of the standard model consists of a three family replication of the representations

$$\begin{aligned} &\text{left-handed fermions:} \quad (3,2)^L_{\frac{1}{6}} \quad (1,2)^L_{-\frac{1}{2}} \\ &\text{right-handed fermions:} \quad (3,1)^R_{\frac{2}{3}} \quad (3,1)^R_{-\frac{1}{3}} \quad (1,1)^R_{-1}. \end{aligned} \tag{9.66}$$

In computing the triangle diagram we have 10 possibilities depending on which factor of the gauge group SU(3)× SU(2)× U(1)$_Y$ appears in each vertex:

SU(3)^3	SU(2)^3	U(1)^3
SU(3)^2 SU(2)	SU(2)^2 U(1)	
SU(3)^2 U(1)	SU(2) U(1)^2	
SU(3) SU(2)^2		
SU(3) SU(2) U(1)		
SU(3) U(1)^2		

It is easy to verify that some of them do not give rise to anomalies. For example, the anomaly for the SU(3)^3 case cancels because left and right-handed quarks transform in the same representation. In the case of SU(2)^3 the cancellation happens term by term using the Pauli matrices identity $\sigma_j\sigma_k = \delta_{jk} + i\varepsilon_{jk\ell}\sigma_\ell$ leading to

$$\mathrm{Tr}\left[\sigma_i\{\sigma_j,\sigma_k\}\right] = 2\,(\mathrm{Tr}\sigma_i)\,\delta_{jk} = 0. \tag{9.67}$$

The hardest condition comes from the three U(1)'s. In this case the absence of anomalies within a single family is guaranteed by the nontrivial identity

$$\begin{aligned} \sum_{\text{left}} Y_+^3 - \sum_{\text{right}} Y_-^3 &= 3\times 2\times\left(\frac{1}{6}\right)^3 + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3 \\ &\quad - 3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \left(-\frac{3}{4}\right) + \left(\frac{3}{4}\right) = 0. \end{aligned} \tag{9.68}$$

It is remarkable that the anomaly exactly cancels between leptons and quarks. Notice that this result holds even if a right-handed sterile neutrino is added since such a particle is a singlet under the whole standard model gauge group and therefore does not contribute to the triangle diagram. We see how the matter content of the standard model conspires to yield a consistent quantum field theory.