Model of the multi-level laser

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Abstract. The laser characteristics depend on the energy-level diagram. A reasonable energy-level diagram is computed to design an optimal laser with high efficiency, noiseless. We present the model of five-level laser. Some laser characteristics such as relative population inversion, pumping threshold, quantum efficiency are investigated. The n-level laser model is possible presented.

I. INTRODUCTION

Level diagram and level diagram selection play a very important role in the theory and in operation of laser. A suitable level diagram selection gives many advantages for laser, such as: high output, no interference, monochromatic pump is not needed.

In this article, we propose a laser which works on n energy level laser.

The symbols used in this article:

- \( i \): \( i^{th} \) energy level (\( i = 1, 2, 3, 4 \ldots \))
- \( n_i \): The number of atoms having \( i^{th} \) energy level in a volume unit of active medium.
- \( N \): Total number of atoms in a volume unit of active substance taking part in laser kinetic process.

- \( X_{mn}^c \): Probability of radiation transition due to induction from (m) level to (n) level (\( X_{mn}^c = X_{nm}^c \)).
- \( X_{mn}^b \): Probability of non-radiation transition from (m) level to (n) level. (\( X_{mn}^b = X_{nm}^b \)).
- \( X_{mn} \): Probability of transition from (m) level to (n) level due to other reasons, such as: auto-transition, transition due to scattering …
- \( H_m \): Absorption probability of atom from basic level (level 1) to level m (\( F_m \) can be considered as pumping rate of laser) and \( H_m \) is proportional to pumping energy.
- \( G_i \): \( i^{th} \) level statistical weight number.
- \( \Delta n \): Population inversion density between 2 levels of laser.
- \( \alpha \): Relative population inversion.
- \( h \): Planck constant.
- \( Q \): Quality of resonator.
- \( G(\nu) \): Renormalization function, a specific characteristic of radiation or absorption spectrum broadening.
- \( \nu_{mn} \): Light frequency between level m and level n according to Bohr axiom.

It is proven that the condition to have laser radiation is: [1]

Generation threshold condition:
\[ \Delta n \geq \frac{1}{h \cdot Q \cdot g(\nu) \cdot R_{mn}^\circ} = \Delta n_0 \quad \text{or:} \quad \frac{\Delta n}{N} \geq \Delta n_0 \cdot \alpha \]

(1)

II. THREE-LEVEL LASER AND FOUR-LEVEL LASER

For a convenient comparison, we introduce the calculating results for three-level laser and four-level laser (Fig.1 and Fig.2).

II.1 Three-level laser

* Relative population inversion:

\[ \Delta n = n_3 - \frac{g_2}{g_1} n_1 = n_2 - K n_1 \]  

(2)

* So:

\[ \frac{\Delta n}{N} = \frac{H_4 \cdot [K X_{31} + (K - 1) X_{21}^c]}{H_4 (1 + X_{32}^c + X_{32}^e) + 2 X_{21}^c + X_{21}} \]  

(3)

* Pumping threshold condition: from (1) and (3) we have:

\[ H_3 \geq H_{ng3} = \frac{\alpha(X_{21} + 2X_{21}^c) + K X_{21}^c + (K - 1) X_{21}^c}{1 - \alpha(1 + X_{21}^c + X_{21})} \]  

(4)

Putting: \( a_1 = X_{21} + 2X_{21}^c \); \( b_1 = K X_{21} + (K - 1) X_{21}^c \); \( c_1 = 1 + \frac{X_{21} + X_{21}^c}{X_{32}} \) leads to:

\[ H_{ng3} = \frac{a_1 + b_1}{1 - ac_1} \]

(5)

II.2 Four-level laser

* Relative population inversion:

\[ \frac{\Delta n}{N} = \frac{H_4 \cdot [K X_{32} + (K - 1) X_{32}^c]}{H_4 (1 + X_{32}^c + X_{32}^e) + 2 X_{32}^e + X_{32}} e^{\frac{hv_{32}}{K T}} \]  

(6)

* Putting:

\( a_2 = [K X_{32} + (K - 1) X_{32}^c] e^{\frac{hv_{32}}{K T}} \)

\( b_2 = (2X_{32}^c + X_{32}) e^{\frac{hv_{32}}{K T}} + (X_{32}^c + X_{32}) \)

\( c_2 = 1 + \frac{X_{32} + X_{32}^c}{X_{43}} \)

leads to pumping threshold:

\[ H_{ng4} = \frac{a_2 + b_2}{1 - ac_2} \]

(7)
III. FIVE-LEVEL LASER

III.1 Working schema

The working schema of five-energy level laser is shown in Fig. 3. Level (1) is the basic level; levels (4) and (5) are excited levels and have relative large level widths. Level (3), (4) and (5) are very close together. Level (1) and (2) are very close together. The laser activity is based on the population inversion between level (3) and level (2):

\[ \Delta n = n_3 - \frac{g_a}{g_b} n_2 = n_3 - Kn_2 \]

In Fig. 3 we do not show probabilities such as: \( X_{42}^c, X_{41}^c, X_{42}^b, X_{41}^b, X_{42}, X_{41}, X_{21}, X_{21}^c \) because \( X_{42}^c, X_{41}^c, X_{42}^b, X_{41}^b, X_{42}, X_{41} \ll X_{23}^b, X_{21}, X_{21}^c \ll X_{21}^b \) and it is noted that: \( X_{32}^b = 0 \).

III.2 Working principle

Atoms in basic level (1) are pumped to the excited levels (4) and (5). Levels (4) and (5) have quite large width levels so the pump is not necessarily monochromatic. Because levels (4) and (5) are unstable, atoms stay there for a short time, after that the non-radiative transitions of atoms from levels (4) and (5) to level (3) occur (level (3) is very close to levels (4) and (5), the transition probability from level (5) to level (4) is too small, so we neglect it). Level (3) is considered as the super-stable level. Level (2) is chosen satisfying the condition: if a transition of atoms from levels (3), (4) and (5) to level (2) (due to induction and other random reasons…) occurs, these atoms will move immediately to the basic level (1) (level (1) is chosen very close to level (2)). After a time interval we have a population inversion between levels (3) and (2) satisfying the laser generate necessary condition.

III.3 Working condition of five-energy level laser.

- Level (3) is super stable \( X_{32}^b = 0 \).
- Level (3) is very close to levels (4) and (5):

\[ X_{43}^b, X_{53}^b \gg X_{43}^c, X_{53}^c, X_{43}, X_{53}. \]

- Level (1) is very close to levels (2): \( X_{21}^b = X_{12}^b \gg X_{21}^c, X_{21} \).

III.4 Calculation for five-energy level laser

The calculation for laser levels is based on the setting up of equilibrium equations for each level between the number of atoms (in a volume unit of active medium) moving to the level (to increase the population) and the number of atoms leaving the level (to decrease the population). Using (+) sign for atoms which increase the population and (-) sign for atoms which decrease the population, denoting \( \frac{dn_i}{dt} \) the atom number variation in level (i), with familiar symbols, we have the following equilibrium equations:
Level (5): \[ \frac{dn_5}{dt} = n_1 H_5 - n_5 X_{35}^b \] (8)

Level (4): \[ \frac{dn_4}{dt} = n_1 H_4 - n_4 X_{43}^b \] (9)

Level (3): \[ \frac{dn_3}{dt} = n_4 X_{43}^b + n_5 X_{35}^b + n_2 X_{23}^c + n_3 (X_{32}^c + X_{32}) \] (10)

Level (2): \[ \frac{dn_2}{dt} = n_3 (X_{32}^c + X_{32}) + n_1 X_{12}^b - n_2 (X_{21}^b + X_{23}^c) \] (11)

The total number of atoms of active medium taking part the laser kinetic process is constant:

\[ n_1 + n_2 + n_3 + n_4 + n_5 = N = \text{const} \] (12)

Levels (2) and (1) satisfy Boltzmann distribution:

\[ n_2 = n_1 \exp(-\frac{h \nu_{21}}{K T}) \] (13)

At stationary states we have: \[ \frac{dn_i}{dt} = 0 \quad (i = 1, 2, 3, 4, 5) \] (14)

Using stationary condition (14), from (8), (9) and (10) we obtain:

\[ n_1 (H_4 + H_5) + n_2 X_{32}^c - n_3 (X_{32}^c + X_{32}) = 0 \] (15)

Noting that: \[ X_{mn}^c = X_{nm}^c, \quad X_{mn}^b = X_{nm}^b \], from (13) and (15) we have:

\[ n_3 = \frac{H_5 + H_4 + X_{32}^c \frac{h \nu_{21}}{K T}}{X_{32}^c + X_{32}} n_1 \]

So: \[ \Delta n = n_3 - kn_2 = n_1 \left[ \frac{H_5 + H_4 + X_{32}^c \frac{h \nu_{21}}{K T}}{X_{32}^c + X_{32}} - Ke^{\frac{h \nu_{21}}{K T}} \right] \] (17)

On the other hand, from (8), (9), (12), (13) and (16) one has:

\[ N = n_1 (1 + e^{\frac{h \nu_{21}}{K T}} + \frac{H_2 + H_4 + X_{32}^c \frac{h \nu_{21}}{K T}}{X_{32}^c + X_{32}} + \frac{H_4}{X_{43}^b} + \frac{H_5}{X_{32}^b}) \] (18)

From (17) and (18) we have the formula for relative population inversion:

\[ \Delta n = \frac{N (H_5 + H_4) - \left[ KX_{32} + (k-1)X_{32}^c \right] e^{\frac{h \nu_{21}}{K T}}}{H_4 (1 + X_{32}^c + X_{32}) + H_4 (1 + \frac{X_{32}^c + X_{32}}{X_{43}^b}) + (X_{32}^c + X_{32}) + (X_{32}^c + X_{32}) e^{\frac{h \nu_{21}}{K T}}} \] (19)

In order to obtain the expression for pump threshold, we introduce the coefficient \( \beta \):

\[ \beta = \frac{H_4}{H_5} \quad (\beta > 0) \] (20)

Putting: \( H = H_4 + H_5 \)

leads to:

\[ H_4 = \frac{\beta H}{1+\beta}; \quad H_5 = \frac{H}{1+\beta} \] (21)

Where \( H \) is pump rate coefficient of atoms from the basic level (1) to upper level (4) and (5). With the notations:

\[ a_3 = \left[ KX_{32} + (k-1)X_{32}^c \right] e^{\frac{h \nu_{21}}{K T}} \]
\[ b_3 = (X_{32} + X_{32}^b) + (X_{32} + 2X_{32}^b) e^{\frac{h\nu_1}{KT}} \]  

\[ c_3 = \frac{\beta}{1 + \beta} \left( X_{32} + X_{32}^c \right) + \frac{\beta}{1 + \beta} \left( \frac{X_{32} + X_{32}^c}{X_{33}^b} \right) \]

we obtain the pump limit expression like that of three-level and four-level laser:

\[ H \geq H_{ng5} = \frac{a_1 + b_1}{1 - a c_3} \]  

Finally, we find the expression for quantum efficiency \( \eta \) of five-level laser (quantum efficiency is the ratio between the energy of generated laser ray and the pump energy):

\[ \eta = \frac{n_3 X_{32}^c v_{32}}{n_1 (H_4 v_{41} + H_5 v_{51})} = \frac{(H_5 + H_4 + X_{32}^c e^{\frac{h\nu_1}{KT}}) X_{32}^c v_{32}}{(H_4 v_{41} + H_5 v_{51}) (X_{32} + X_{32}^c)} \]

From (21) we have:

\[ \eta = \frac{X_{32}^c}{X_{32} + X_{32}^c} \left( \beta + 1 \right) H (H + X_{32}^c e^{\frac{h\nu_1}{KT}}) v_{32} \]  

### III.5 Evaluation and discussion

Expression (19) shows that when some small transitions are neglected, the relative population inversion density depends only on \( X_{32}^c, X_{32} \) and \( X_{43}^b, X_{53}^b \). The dependence of \( (\Delta n/N) \) on \( F \) is showed in Fig. 4.

Fig. 4 shows that there is a limit of relative population inversion. This limit equals \( \frac{1}{c_3} \).

In physical side, it means that although many atoms are pumped to level (3), there always exist induction radiations which lead to the relaxation of level (3).

Note that in practice \( (\Delta n/N) \) depends not only on \( F \) but also on external factors such as, thermal conductivity or mechanical durability of high active medium…

- He dependence of pump limit \( H_{ng5} \). On \( \alpha \) is showed in Fig 5. If \( \alpha \) becomes large, pump limit also becomes large. Formula (23) shows that to save energy, we can chose the following cases:

a. \( \frac{h\nu_1}{KT} \) large. But this cause the following obstacles: level (2) is quite far from level (1) (which do not satisfy the working condition), low productivity…

b. \( X_{41}^b, X_{53}^b \gg X_{32}^b, X_{32} \): It means that levels (4) and (5) are very close together. This is realizable in practice.
Note that from (19), if putting $H_5 \rightarrow 0 \Rightarrow H_4 \rightarrow H$ we obtain (16). So that four-energy level laser is only a special case of five-energy level laser.

Five-level laser has two advantages:

a. Reducing considerably the diffraction so that the generated laser rays are highly monochromatic.

b. Not requiring too monochromatic pump light. This will clarify in Section 6.

### III.6 N-level laser

The schema of n level laser is plotted in Fig.6. Levels 4, 5, 6,..., n are excited. Level 1 is the basic level. The working of the laser is based on level (3), (2) (or certain p, q) satisfying population inversion condition (condition (1)). Doing the calculation work like that of five-level laser leads to the following result (for laser whose working is based on 2 levels (2) and (3)).

- $\Delta n = \frac{\sum_{k=4}^{n} H_k \cdot [KX_{32} + (K-1)X_{32}^e] e^{\frac{hv_{31}}{KT}}}{\sum_{k=4}^{n} H_k \left(1 + \frac{X_{32} + X_{32}^e}{X_{32}^b} + (X_{32} + X_{32}^e) + (X_{32} + X_{32}^e) e^{\frac{hv_{31}}{KT}}\right)}$ (25)

N-level laser has quite large pump light spectrum widths, depending on the number of level n. The defect of n level laser usually is its efficiency is lower than that of three-level or four-level laser.

### References


