”Superconformal Algebra and
HyperKähler Manifolds

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hep-th/

♠ N=4 Superconformal Algebra

We consider theories based on N=4 superc

bra (SCA). N=4 SCA contains a level-k SU(2) when $c = 6k$. There exist both BPS and nor

tations in these theories. In the case of $k =$

less) representation of isospin=0 the charact

the form

$$ch^{\tilde{R}}_{k=1,\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau$$

where

$$\mu(z;\tau) = \frac{ie^{\pi iz}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e}{1-q^n e}$$

Function $\mu(z;\tau)$ is a typical example of the s

theta functions (Lerch sums). Mock theta fun

theta functions but they do not have good m
ties. In general character of BPS (massless)
have non-trivial denominators due to BPS con
0) and are thus Mock theta functions. They in
have good modular behavior.

The above function μ has the S-tranformation

$$\mu(z;\tau)+\sqrt{\frac{i}{\tau}}\mu(\frac{z}{\tau};-\frac{1}{\tau})=\frac{1}{2}M$$

$$M(\tau)=\int_{-\infty}^{+\infty}\frac{e^{\pi i\tau z^2}}{\cosh\pi z}dz$$

$$\Uparrow$$

Modell integral

Recently there has been a development in unc
nature of Mock theta functions by Zwegers anc
oped a way to improve their modular prope
prescription to cure the modular property is to
a partner of $\mu(z;\tau)$

$$R(\tau) = \sum (-1)^n [sgn(n+\frac{1}{2}) - E((n+$$

$$\times q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \quad \tau = u$$

Here E denotes the error function. By construc

a transformation law

$$R(\tau)+\sqrt{\frac{i}{\tau}}R(-\frac{1}{\tau})=M(\tau)$$

Then we form a combination

$$\hat{\mu}(z;\tau)\equiv\mu(z;\tau)-\frac{1}{2}R(\tau)$$

The Mordell integral cancels out and $\hat{\mu}$ has a

property

$$\hat{\mu}(z;\tau)=-\sqrt{\frac{i}{\tau}}\hat{\mu}(\frac{z}{\tau};-\frac{1}{\tau})$$

and is in fact a Jacobi form. Explicitly R is e

contour integral

$$R(\tau) = -i \int_{-\bar{\tau}}^{i\infty} \frac{\eta(z)^3}{\sqrt{\frac{z+\tau}{i}}} dz$$

Now consider

$$\begin{aligned} J(z,w;\tau) &= \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \left(\hat{\mu}(z;\tau) - \hat{\mu}(w \right. \\ &= \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \left(\mu(z;\tau) - \mu(w \right. \\ &= ch^{\tilde{R}}_{k=1,\ell=0}(z\tau) - \mu(w;\tau \end{aligned}$$

By construction

$$J(w,w;\tau)=0$$

Together with the transformation properties o

can determine them as

$$\begin{aligned}
J(z,w=\frac{1}{2};\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right. \\
J(z,w=\frac{1+\tau}{2};\tau) &= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right. \\
J(z,w=\frac{\tau}{2};\tau) &= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right.
\end{aligned}$$

We then find

$$ch^{\tilde{R}}_{k=1,\ell=0}(z;\tau) = \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + h_2($$

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where

$$h_2(\tau) = \frac{1}{\eta(\tau)}\mu(\frac{1}{2};\tau), \quad h_3(\tau) = \frac{1}{\eta(\tau)}$$

$$h_4(\tau) = \frac{1}{\eta(\tau)}\mu(\frac{\tau}{2};\tau)$$

♣ Elliptic genus of K3

$$Z_{elliptic}(z;\tau) = Tr_{R\times R}(-1)^{F_L+F_R}e^{2\pi i z J^3_{L,0}}$$

Elliptic genus of K3 surface is given by: E

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 \right.$$

$$Z_{K3}(z=0) = 24, \quad Z_{K3}(z=\frac{1}{2}) = 1$$

$$Z_{K3}(z=\frac{1+\tau}{2}) = 2q^{-\frac{1}{2}} + O(q^{\frac{1}{2}})$$

$$Z_{K3} = 24ch^{\tilde{R}}_{k=1,\ell=0}(z;\tau) - 8\sum_i h(\tau)$$

Note:

non-BPS (massive) representations of level k N

$$q^{h-\frac{(\ell+\frac{1}{2})^2}{k+1}-\frac{k}{4}}\;\chi_{k-1,\ell}\;\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

At unitarity boundary ($h=k/4$) it splits into m

$$q^{-\frac{(\ell+\frac{1}{2})^2}{k+1}}\;\chi_{k-1,\ell}\,\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}=ch^{\tilde{R}}_{k,\ell+\frac{1}{2}}+2ch^{\tilde{R}}_{k,}$$

q-expansion

$$8\eta(\tau)(h_2(\tau)+h_3(\tau)+h_4(\tau))=\qquad 2q^{-\frac{1}{8}}$$

$$\Uparrow$$

polar term

In the present case $k = 1, \ell = 0$,

$$q^{-\frac{1}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} = ch^{\tilde{R}}_{k=1,\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{k=1}$$

and

$$Z_{K3} = 20ch^{\tilde{R}}_{k=1,\ell=0} - 2ch^{\tilde{R}}_{k=1,\ell=\frac{1}{2}} + \sum_{n=1} A_n$$

Using the method of Rademacher expansion

mine the asymptotic behavior of coefficients

$$A_n \approx e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}}$$

Let us denote the massless rep. of isospin ℓ

partition function has the form

$$Z_{pf}=20|[0]|^2+|\,[1/2]\,|^2+....$$

in Ramond sector. Under the spectral flow $[$

tween NS and R sectors and we have coeff

identity representation.

♠ Generalization to higher levels

Level-k N=4 SCA describes complex-2k dimen

manifolds. Typical examples are symmetric

$K3^{[n]}$. Building blocks of level-k theory are

$$Z_{X_k^{(1)}} : \Big[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)} \right)^{2k} + (2 \to 3) + (3$$

$$Z_{X_k^{(2)}} : \Big[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)} \right)^{2k-2} \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)} \right)^{2} +$$

and other symmetric polynomials of the

Ellptic genera of hyperKähler manifolds are ob bining these building blocks: in a sense they by the leading term $Z_{X_k^{(1)}}$. Only this term cont representation (identity rep. in NS sector) an normalization by demanding the unit coeffic

function.

The only contribution to $\hat{A}$ genus comes fro

term and one can immediately derive

$$\hat{A} = k + 1$$

for any hykerKähler manifolds in $2k_{\mathrm{C}}$-dimen

Sawon, 2001.

Ellitpic genus of Hilbert scheme of points of K etc is given by

$$48 Z_{X_2^{(1)}} + 60 Z_{X_2^{(2)}}$$

Entropy of hyperKähler manifolds H_{2k}

Ellitpic genus (H_{2k})

$$= \sum_{I=0}^{k/2} n_I [I] + \sum_{\ell} \sum_{n} A^{\ell}_{n} q^{n-\alpha_\ell} \chi_{k-1,}$$

Asymptotic growth of A^{ℓ}_{n} is completely dete

polar parts of the series $\{A^{\ell'}_{m}\}$.

$$\boldsymbol{A_n^{\ell} \approx \exp(\pi\sqrt{4kn - \ell^2})}$$